Lectures on Public Goods Mechanisms

The Vickrey, Clark, Groves Mechanism

Remember our discussion of the pivotal mechanism. The government chooses the outcome that maximizes the sum of reported willingness to pay and collects nothing from an individual i if his report doesn't change the outcome. If it does shift the outcome, then i pays an amount equal to the difference between amounts of the two sides not counting his own report. This amount is exactly the difference $X - Y$ between the summed value X to the rest of the population of the outcome when i's answer is not counted and the summed value Y to the rest of the population if i's value is counted. This same principle can be applied in cases where the government's choice can be any positive number rather than just a binary choice.

Costless Public Goods

We first consider the special case where public goods are costless, but people still disagree about how much should be provided. Suppose that there are n people who share consumption of a public good. Each person i has initial wealth w_i and a quasilinear utility function $U(x, y) = x + u_i(y)$ and where u_i is a single-peaked function that increases for $y \leq y_i^+$ and decreases for $y \geq y_i^+$.

Step 1: All players are asked to report their utility functions to the authority. Let $m_i(\cdot)$ be the function reported by i. Where the vector of reported utility functions is given by $m = (m_1, \ldots m_n)$, the central authority chooses $\bar{y}(m)$ to maximize the sum of the reported utility functions.

Step 2: The central authority calculates a tax to be assessed to each individual. The tax $T_i(m)$ assessed on person i is equal to the difference between the maximum sum of utilities that the other players could achieve if Player i played no part in the decision and the sum of the utilities that the other players achieve, given the message that Player i sends. More specifically, the maximum sum of utilities that the others could achieve while ignoring i's signal is

$$Z_{\sim i}(m) = \max_y \left\{ \sum_{j \neq i} m_j(y) \right\}.$$

The sum of the utilites that the others achieve when Player i's message is used to determine $\bar{y}(m)$ is

$$Y_{\sim i}(m) = \sum_{j \neq i} m_j(\bar{y}(m)).$$

Thus we let $T_i(m) = Z_{\sim i}(m) - Y_{\sim i}(m)$.

Step 3: The quantity $\bar{y}(m)$ is chosen. Each player i pays an amount

$$T_i(m) = Z_{\sim i}(m) - Y_{\sim i}(m) \geq 0.$$

This revenue is not reimbursed to the players.

Note the following:

- The amount $Z_{\sim i(m)}$ defined in Step 3 does not depend on i's reported utility function.

- Since $Z_{\sim i}(m) = \max_y\{\sum_{j\neq i} m_j(\bar{y}_j)\}$, it must be that $Z_{\sim i}(m) \geq \sum_{j\neq i} m_j\left(\bar{y}(m)\right) = Y_{\sim i}(m)$. It follows that the net tax paid by person i, which is $Z_{\sim i}(m) - Y_{\sim i}(m)$ must be positive. Interpreting this net tax, we see that it is the effect of person i's response on the total utiilty that the other players get from y.

- If the vector of messages sent by all players is m, and the government chooses y, then the after tax income of player i will be

$$w_i - Z_{\sim i}(m) + \sum_{j\neq i} m_j(y).$$

  and so the utility of player i will be

$$w_i - Z_{\sim i}(m) + \sum_{j\neq i} m_j(y) + u_i(y).$$

- Since the message sent by player i's has no effect on either w_i or Z, the only effect of player i's message on his own utility acts through its influence on the amount of public good y that the government chooses. In particular player i would like the government to choose y to make

$$\sum_{j\neq i} m_j(y) + u_i(y)$$

  as large as possible.

- Recall that the government chooses y to maximize

$$\sum_{j\neq i} m_j(y) + m_i(y).$$

  What is the best thing that player i can do for himself?. If he reports $m_i = u_i$, then the government will choose y to maximize

$$\sum_{j\neq i} m_j(y) + u_i(y),$$

  which is exactly what i wants it to do. If i reports $m \neq u$, the government will solve a different maximization problem which in general will result in a lower value of

$$\sum_{j\neq i} m_j(y) + u_i(y).$$

With these results we conclude that

1. reporting one's true utility is a weakly dominant strategy.

2. When everybody uses their weakly dominant strategy, the outcome is the Pareto optimal amount of y.

3. The net tax collected from each person is non-negative.

4. If a positive taxes are collected, the outcome is not efficient, since the mechanism doesn't allow for them to be rebated.

Party animals: an illustrative spcial case:

Three friends, Archie, Betty, and Veronica are planning a party. They disagree about how many people to invite. Each person i has an initial endowment of W_i dollars and a quasilinear utility function of the form

$$u_i(x_i, y) = x_i + a_i y - \frac{1}{2}y^2 \tag{1}$$

where x_i is the number of dollars that i has to spend and y is the number of people invited to the party. They all know that all three have utility functions of this functional form, but only person i knows the parameter a_i from his or her own utility function. For this example, let us suppose that $a_A = 20$, $a_B = 40$, and $a_V = 60$.

Implementing the VCG mechanism we have:

Step 1: Each person is asked to report his or her parameter a_i. They are not necessarily required to tell the truth. Let m_i be the value reported by person i. The friends agree to choose a number of persons y to maximize the sum of the reported utility functions, which is

$$(m_A + m_B + m_V)y - \frac{3}{2}y^2.$$

This happens when

$$y(m) = \frac{m_A + m_B + m_C}{3}.$$

(We have shown that if each person plays his or her best strategy, then $m_i = a_i$ for all i. In this case, the number of persons invited is $y(m) = \frac{a_A + a_B + a_V}{3} = 40$.)

Step 2: Now we calculate the "tax" paid by each player.

First let's find the Z's. For Archie, we have

$$Z_{\sim A}(m) = \max_y \left\{(m_B + m_V)y - y^2\right\}.$$

This is maximized when $y = (m_B + m_V)/2$. A simple calculation then shows that

$$Z_{\sim A}(m) = \frac{(m_B + m_V)^2}{4}.$$

(When Betty report their parameters truthfully, it must therefore be that $m_B + m_V = 100$ and $Z_{\sim A}(m) = 2500$.

Similar calculations show that for Betty, $Z_{\sim B} = \frac{(m_A+m_V)^2}{4}$. Then when Archie and Veronica report their parameters truthfully, $m_A + m_V = 80$, so that $Z_{\sim B}(m) = 1600$.

For Veronica, $Z_{\sim V} = \frac{(m_A+m_B)^2}{4}$. When Archie and Betty report their parameters truthfully, $m_A + m_B = 60$ so that $Z_{\sim V} = 900$.

Now we find the Y's. For Archie,

$$Y_{\sim A} = u_A(\bar{y}(m)) + u_B(\bar{y}(m)) = (m_A + m_B)\bar{y}(m) - y(m)^2.$$

When Betty and Veronica reveal their parameters truthfully, we then have

$$Y_{\sim A} = 100 \times 40 - 40^2 = 2400.$$

For Betty, when Archie and Veronica reveal their parameters truthfully, we have

$$Y_{\sim B} = (m_A + m_V)\bar{y}(m) - y(m)^2 = 80 \times 40 - 40^2 = 1600.$$

For Veronica, when Archie and Betty reveal their parameters truthfully, we have

$$Y_{\sim V} == (m_A + m_B)\bar{y}(m) - y(m)^2 = 60 \times 40 - 40^2 = 800.$$

Now, we can calculate the T's when all three reveal their parameters truthfully.

For Archie, $T_A = Z_A(m) - Y_A(m) = 2500 - 2400 = 100$. Fof Betty, $T_B = 1600 - 1600 = 0$. For Veronica, $T_V = 900 - 800 = 100$.

Step 3: Putting it all together, we find that they have a party with 40 guests. Archie has to pay a tax of 100, Betty pays a tax of 0 , and Veronica has to pay 100. Thus the mechanism achieves a Pareto efficient party size, but wastes 200.

Stated informally, Archie's report of his utility function makes for a smaller party than Betty and Veronica would choose, while Veronica's report makes for a larger party than Archie and Betty would choose. Betty's report doesn't change the size of party that the other two would choose. Those who change the party size, must pay the others for the "inconvenience" that they impose.

What if there are more hosts?

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Suppose that the party is organized not by just 3 people, but by a dormitory floor with 21 residents. All of these residents have utility functions of the same form, as those of Archie, Betty, and Veronica. Seven of them have $a_i = 20$, seven have $a_i = 40$ and seven have $a_i = 60$. What happens now with the VCG?

If we follow the rules of the VCG, each player submits his or her alleged value m_i of the parameter a_i. If all use their weakly dominant strategies, each will report $m_i = a_i$. The party size $y(m)$ will then be chosen to maximize

$$(\sum_i a_i)y - \frac{21}{2}y^2.$$

This is maximized when

$$y(m) = \frac{\sum_i a_i}{21} = 40.$$

If Person i has $a_i = 20$, then the sum of the utilities that the other 20 residents get from the party is

$$Y_{\sim i}(m) = (6 \times 20 + 7 \times 40 + 7 \times 60)40 - 10 \times 40^2 = 32,800 - 16000 = 16,800.$$

The sum of the utility of the other 20 residents would be maximized with $y = 41$ and takes the value $(6 \times 20 + 7 \times 40 + 7 \times 60)41 - 10 \times 41^2 = 16,810$. Thus in equilibrium, each of the residents with $a_i = 20$ would have to pay a net amount of $16810 - 16800 = 10$.

Similar calculations show that the residents with $a_i = 40$ would pay 0 and the residents with $a_i = 60$ would each pay 10. The party would be of the Pareto optimal size, 40. In this case, the total waste from the mechanism is $14 \times 10 = 140$.

Question: What is the amount of waste if you have $3N$ hosts deciding, with N of each of the three types described above. What happens in the limit as N gets large?

Costly Public Goods

What if the public good is costly? Suppose that the total cost of providing y units of public good is given by an increasing convex function $C(y)$. When the public good is costly, we need a mechanism that will collect at least as much tax revenue as is needed to pay for the amount of public good chosen. When preferences are quasi-linear, the method used for costless public goods can be applied with a simple modification. The trick is to suppose that in addition to the VCG taxes $T_i(m)$ described in the previous section, each player must pay a tax $\frac{1}{n}C(y)$ when the amount of public good is y.

Where the utility function of Player i is $U(x_i, y) = x_i + u_i(y)$, let us define $v_i(y) = u_i(y) - \frac{1}{n}C(y)$ and define the function $V(x_i, y) = x_i + v(x_i, y)$. Notice that if u_i is concave and C is convex, then v_i is a concave function. It is then reasonable to assume that v_i is an increasing function for small values of y and decreasing when y is sufficiently large.

Now the VCR can be applied directly for players with utility functions $V(x_i, y)$. The central authority announces the cost function $C(y)$ to all players. Player i is asked to report $v_i(y) = u_i(y) - \frac{1}{n}C(y)$. Where $m_i(y)$ is the message from Player i, the central authority chooses $\bar{y}(m)$ to maximize $\sum_i m_i(y)$. If players all report their functions truthfully, this means that $\bar{y}(m)$ maximizes

$$\sum_i v_i(y) = \sum_i u_i(y) - C(y)$$

, which is the total consumers' surplus from y. The central authority then computes VCG taxes $T_i(m) \geq 0$ for each player, just as in the previous discussion. The total tax bill for Player i will be $T_i(m) + \frac{1}{n}C(y)$ and total revenue will be

$$\sum_i \left(T_i(m) + \frac{1}{n}C(y) \right) = \sum_i \left(T_i(m) \right) + C(y) \geq C(y).$$

This same exercise will work with predetermined costs shares that are not equal. In the previous discussion, simply replace the share $1n$ for Player i with $\theta_i \geq 0$ such that $\sum_i \theta_i = 1$.

Groves-Ledyard Mechanism

The Groves-Ledyard mechanism has a property that in Nash equilibrium an efficient amount of public goods is produced and taxes exactly pay for this amount. Public goods are produced at a constant unit cost c. (This assumption could be generalized.) Each player is asked to state an amount m_i that he or she would like to add to the amount of public goods supplied. Let

$$\bar{m} = \frac{1}{n} \sum_{i=1}^{n} m_i$$

The total amount of public goods supplied will be

$$y = \sum_{i=1}^{n} m_i = n\bar{m}.$$

Let

$$\bar{m}_{\bar{i}} = \frac{\sum_{j \neq i} m_j}{n-1}$$

and let

$$R_i(m) = \frac{1}{n-2} \sum_{j \neq i} (m_j - \bar{m}_{\bar{i}})^2.$$

If the vector of messages sent by players is $m = (m_1, \ldots, m_n)$, each player will pay a tax equal to

$$\frac{c}{n} + \frac{\gamma}{2} \left(\frac{n-1}{n} (m_i - \bar{m}_i)^2 - R_i(m) \right)$$

The functions $R_i(m)$ have been chosen to have two properties. The first is that $R_i(m)$ does not depend i any way on m_i. The second is that

$$\sum_{i=1}^{n} R_i(m) = \frac{n-1}{n} \sum_{i=1}^{n} (m_i - \bar{m}_i)^2.$$

In this case, player i's private consumption will be

$$x_i(m) = W_i - \frac{c}{n} - \frac{\gamma}{2}\left(\frac{n-1}{n}(m_i - \bar{m}_i)^2 - R_i(m)\right)$$

and the amount of public goods supplied will be $y(m) = \sum_{i=1}^{n} m_i$. Therefore consumer i's utility with this vector of messages will be $U^i(x_i(m), y(m))$. The derivative of player i's private consumption with respect to m_i is then equal to

$$\frac{c}{n} + \gamma\frac{n-1}{n}(m_i - \bar{m})\left(1 - \frac{1}{n}\right) = \frac{c}{n} + \gamma(m_i - \bar{m}).$$

The first order condition for maximizing player i's utility with with respect to m_i is then seen to be that i's marginal rate of substitution between public goods and private goods is just

$$MRS_i = \frac{c}{n} + \gamma(m_i - \bar{m}).$$

This section needs to be repaired. It is not so interesting to show incentive compatibility again in this special case. More interesting to calculate net revenue collected by mechanism and perhaps to add costly production.

A more general quadratic example

Suppose that each of n consumers has utility function

$$u_i(x_i, y) = x_i + a_i y - \frac{1}{2}y^2$$

and the message m_i sent by consumer i is i's report of his parameter a_i.

Step 1: Where the vector of reported utility parameters is given by m, the central authority chooses y to maximize the sum of the reported utility functions. That is, it chooses y to maximize

$$\sum_{i=1}^{n}\left(m_i y - \frac{1}{2}y^2\right) = y\sum_{i=1}^{n} m_i - \frac{n}{2}y^2. \tag{2}$$

Setting the derivative with respect to y equal to zero, we see that this sum is maximized when $y = y(m)$ where

$$y(m) = \frac{1}{n}\sum_i m_i. \tag{3}$$

Step 2: The central authority makes a sidepayment to each i that is equal to the sum of the reported utilities of y for all other persons. For any person i, this means that person i gets a sidepayment equal to

$$y\sum_{j\neq i} m_j - \frac{n-1}{2}y^2 \tag{4}$$

Step 3: The central authority collects from each person, an amount equal to the maximum possible sum of the reported utilities of the other persons. For any person i, this amount is equal to

$$\max_y\{y \sum_{j \neq i} m_j - \left(\frac{n-1}{2}\right) y^2\}. \tag{5}$$

Setting the derivative equal to zero, we see that this sum maximized when

$$y = \frac{\sum_{j \neq i} m_j}{n-1}$$

and hence the maximal possible sum of utilities for the other persons is

$$\frac{1}{2(N-1)} \sum_{j \neq i} m_j^2.$$

Solving Person i's decision problem:

Where the number of persons attending the party determined by the mechanism, given the vector of responses m, is $y(m)$, when we take account of the payments in steps 2 and 3, Person i will have consumption $x_i(m)$ of other goods where

$$x_i(m) = W_i + y(m) \sum_{j \neq i} m_j - \frac{n-1}{2} y(m)^2 - \frac{1}{2(N-1)} \sum_{j \neq i} m_j^2. \tag{6}$$

Given that i's utility function is $u_i(x_i(m), y(m)) = x_i(m) + a_i y(m) - \frac{1}{2} y(m)^2$, we can calculate i's utility. In particular to maximize his utility, i will choose m_i that maximizes this utility, which is equal to

$$u_i(x_i(m), y(m)) = x_i(m) + a_i y(m) - \frac{1}{2} y(m)^2. \tag{7}$$

To find the response m_i that maximizes his utility, i would set the partial derivative of Equation **??** with respect to m_i equal to 0. This implies that

$$\frac{\partial x_i(m)}{\partial m_i} + a_i \frac{\partial y(m)}{\partial m_i} - y(m) \frac{\partial y(m)}{\partial m_i} = 0. \tag{8}$$

From Equation **??** we have

$$\frac{\partial y(m)}{\partial m_i} = \frac{1}{n}. \tag{9}$$

From Equations **??** and **??** it follows that

$$\begin{aligned} \frac{\partial x_i(m)}{\partial m_i} &= \left(\sum_{j \neq i} m_j\right) \frac{\partial y(m)}{\partial m_i} - (n-1) y(m) \frac{\partial y(m)}{\partial m_i} \\ &= \frac{1}{n} \sum_{j \neq i} m_j - \frac{n-1}{n} y(m) \end{aligned} \tag{10}$$

Then from substituting from **??** and **??** into **??** we have

$$\frac{1}{n}\left(a_i + \sum_{j\neq i} m_j\right) - y(m) = 0 \tag{11}$$

It follows from Equation **??** and **??** that

$$\frac{1}{n}\left(a_i + \sum_{j\neq i} m_j\right) - \sum_{j=1}^{n} m_j = 0 \tag{12}$$

which implies that $m_i = a_i$. Thus we have demonstrated that whatever numbers the others claim describe their utility functions, the best response for person i is to announce his true value $m_i = a_i$.

Since this reasoning applies for every individual i, it must be that announcing $m_i = a_i$ is a dominant strategy for every individual. Hence if all play their dominant strategies, the quantity of y selected will be the Pareto efficient quantity $\frac{1}{n}\sum_{i=1}^{n} a_i$ that maximizes the sum of utilities.

Groves Ledyard remarks

Let us try applying the Groves-Ledyard mechanism to the same n person society. In the Groves Ledyard mechanism, each player reports a number m_i and the size chosen for the party is the sum of these numbers. Define the mean of the reported numbers to be

$$\mu = \frac{1}{N}\sum_{j=1}^{n} m_j$$

and define

$$\mu_i = \left(\frac{1}{N-1}\right)\sum_{j\neq i} m_j$$

to be the mean of the numbers named by persons other than i. Define σ_i^2 to be the variance of the numbers submitted by persons other than i. If the vector of numbers submitted is m, each player i will pay a net tax

$$T_i(m) = \frac{\gamma}{2}\left(\frac{N-1}{N}(m_i - \mu_i)^2 - \sigma_i^2\right).$$

With this tax scheme and with the size of party being $x = \sum_i m_i$, the utility of person i will be

$$W_i + a_i \sum_i m_i - \left(\sum m_i\right)^2 - T_i(m) \tag{13}$$

Person i will find his or her best choice of m_i by setting the derivative with respect to m_i of Expression **??** equal to zero. This happens when

$$a_i - \sum m_i - \frac{\partial T_i(m)}{\partial m_i} = 0$$

or equivalently when

$$a_i - \sum m_i = \gamma \left(\frac{N-1}{N} \left(m_i - \mu_i \right) \right) \tag{14}$$

A bit of simple algebra[1] shows that

$$\frac{N-1}{N} \left(m_i - \mu_i \right) = m_i - \mu$$

and therefore Equation **??** is equivalent to

$$a_i - \sum m_i = \gamma \left(m_i - \mu \right) \tag{15}$$

Summing both sides of Equation **??** over n, we find that

$$\sum_{i=1}^{n} a_i - n \sum_{i=1}^{n} = 0 \tag{16}$$

and therefore

$$\sum_{i=1}^{n} m_i = \frac{1}{n} \sum a_i \tag{17}$$

[1] Proof is as follows:

$$\begin{aligned} m_i - \mu &= m_i - \frac{1}{N} \left(\sum_{j=1}^{n} m_j \right) \\ &= m_i \left(1 - \frac{1}{N} \right) - \frac{1}{N} \sum_{j \neq i} m_j \\ &= m_i \frac{N-1}{N} - \frac{N-1}{N} \mu_i \\ &= \frac{N-1}{N} \left(m_i - \mu_i \right) \end{aligned}$$