Calculus conditions for concave functions (of a single variable).

- Recall that a real-valued function f is concave if and only if its domain is a convex set $A \subset \Re_n$ and for all x_1 and x_2 in A and for all $\lambda \in [0, 1]$,

$$f\left(\lambda x_1 + (1 - \lambda)x_2\right) \geq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

- If $A \subset \Re$, this implies that for all x_1 and x_2 in A and for all $\lambda \in [0, 1]$,

$$f(x_2) \leq f(x_1) + (x_2 - x_1)f'(x_1).$$

- Draw some pictures.–See slide for "rooftop theorem"

From rooftops to second derivatives.

- The rooftop theorem tells us that if f is concave, $f(x_2) \leq f(x_1) + (x_2 - x_1)f'(x_1)$ for all x_1 and x_2 in A.
- Rearranging terms, we have

$$f(x_2) - f(x_1) \leq (x_2 - x_1)f'(x_1). \tag{1}$$

- The rooftop theorem also tells us that if f is concave, $f(x_1) \leq f(x_2) + (x_1 - x_2)f'(x_2)$ for all x_1 and x_2 in A.
- Rearranging terms, we have $f(x_1) - f(x_2) \leq (x_1 - x_2)f'(x_2)$.
- Multiply both sides by -1, above implies

$$f(x_2) - f(x_1) \geq (x_2 - x_1)f'(x_2) \tag{2}$$

- Let $x_2 > x_1$. Then Inequalities 1 and 2 imply that $f'(x_2) \leq f'(x_1)$.

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Maxima for differentiable concave functions

- From elementary calculus we know that $f : \Re \to \Re$ and if $f'(x)$ exists and x is in the interior of its domain, then a necessary condition for x to be a local maximum of f on A is that $f'(x) = 0$.
- We also know that a sufficient condition for x to be an interior local max is that $f'(x) = 0$ and $f''(x) < 0$ and $f''(x) < 0$ for all $x \in A$. Is this condition also necessary?
- Show that if f is a concave function and has a local max at x, then it has a global max at x.
- So we know that if f is a concave function such that $f''(x)$ exists everywhere, the $f'(x) = 0$ is necessary and sufficient for x to be a global maximum on A.
- Is $f'(x) = 0$ and $f''(x) \leq 0$ sufficient for x to be a maximum?

Going to higher dimensions

- Where $f : \Re^n_+ \to \Re$, for all x and y in $\Re^n$, let us define $g(t) = f(x + t(y - x))$ for all $t \in [0, 1]$
- If f is a concave function, then g must be a concave function.
- So it must be that $g''(0) \leq 0$.
- So lets find out more about $g''(0)$.

Producing a quadratic form

- Applying the chain rule,

$$g'(t) = (y_i - x_i) \sum_{i=1}^{n} f_i(x + t(y - x)).$$

- Then

$$g''(t) = \sum_{i=1}^{n} (y_i - x_i) \frac{d}{dt} f_i(x + t(y - x)).$$

- So

$$g''(t) = \sum_{i=1}^{n} (y_i - x_i) \sum_{j=1}^{n} (y_j - x_j) f_{ij} \left(x + t(y - x)\right).$$

Negative semi-definiteness

- We now know that if g is concave, then for all x and y, it must be that

$$g'(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(x) \leq 0.$$

- This will be true if and only if the "Hessian" matrix, whose elements are $f_{ij}(x)$ is negative semi-definite.
- That is the case if its principle minors of order k are negative for odd k and positive for even k.

An example

- Let

$$f(x_1, x_2) = (x_1 + x_2) - \frac{1}{2}(x_1^2 + x_2^2) + cx_1x_2$$