JOURNAL OF ECONOMIC THEORY **68**, 531–543 (1996)
ARTICLE NO. 0029

Notes, Comments, and Letters to the Editor

Cores and Competitive Equilibria with Indivisibilities and Lotteries

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Received December 23, 1993; revised December 15, 1994

Lotteries are introduced in the exchange model of Shapley and Scarf (*J. Math. Econ.* **1** (1974), 23–37). In competition, traders buy and sell probabilities on houses. When considering the core, feasible allocations of the economy and blocking allocations of coalitions may involve lotteries. Among other results, we show that the set of lottery equilibrium allocations is non-empty and does not contain all competitive equilibrium allocations, the core of the Shapley/Scarf model is a strict subset of the lottery core, and lottery equilibrium allocations are contained in the lottery core. *Journal of Economic Literature* Classification Numbers: C62, C71.

1. INTRODUCTION

Shapley and Scarf [12] describe a market that redistributes ownership of indivisible commodities, which they refer to as houses. They consider competitive outcomes and core outcomes, establishing the existence of competitive equilibrium, the non-emptiness of the core, and the relationship of competitive equilibria to the core, among other things.

In this paper, lotteries are introduced in the Shapley/Scarf model. A lottery is a probability distribution over houses. In competition, consumers buy and sell probability on houses to obtain desired lotteries. When considering the core, feasible allocations of the economy and blocking allocations of coalitions may involve lotteries.

* We thank John Marshall, Karl Shell, Bruce Smith, Myrna Wooders, and two anonymous referees for helpful comments. We also thank Lloyd Shapley for comments that prompted us to develop results that are independent of the choice of utility representations of traders' preferences. We are grateful to the Academic Senate at UCSB for financial support.

0022-0531/96 $18.00

The analysis assumes traders are von Neumann–Morgenstern expected utility maximizers. This means traders possess *preference scaling functions* that are unique up to positive affine transformations and determine their preferences over lotteries. However, the analysis also addresses equilibrium and cores when the traders' preference scaling functions are not known. In that case it uses only the ordinal rankings, which are the type of information used by Shapley and Scarf in their analysis.

The existence of lottery equilibrium allocations is established. An example is provided which demonstrates that not all competitive equilibrium allocations for the model without lotteries reappear as degenerate lottery equilibrium allocations once lotteries are introduced. Non-degenerate lottery equilibria exist, but lotteries are somewhat inessential because final utilities do not depend on their outcomes. If no trader is indifferent between any two of the houses, then there is a unique competitive equilibrium allocation for the model without lotteries (Roth and Postlewaite [9]) which can be supported as a degenerate lottery equilibrium allocation.

All core allocations for the Shapley/Scarf model are in the core for the same economy with lotteries. That is, the pure core is contained in the lottery core. Furthermore, the lottery core includes randomizations over pure core allocations where final utilities depend on the outcome of lotteries. If the traders' preference scaling functions are known, then there exist economies for which there are lottery core allocations that are not simply randomizations over pure core allocations. Lottery equilibrium allocations are contained in the lottery core.

Issues that arise from introducing lotteries to a market with a finite number of traders connect our work to previous work by Hylland and Zeckhauser [5] and Garratt [2]. Each of these papers deals with indivisibility and the existence of lottery equilibrium. A previous adaptation of the Shapley/Scarf model was made by Quinzii [8] who introduced money.

2. Shapley/Scarf Exchange Model

There are n traders, $i = 1, 2, ..., n$, each of whom is endowed with one unit of an indivisible good, house i, and nothing else. The set of traders is denoted by N. Houses differ and traders differ in their preferences for them. Trader i's ordinal ranking of the n houses is denoted by $\succcurlyeq_i$. Each trader ranks owning no house below owning any of the houses and owning multiple houses is ranked equal to the highest of their separate ranks. Thus, the sole purpose of the market is to redistribute the n houses among the n traders.

Details of the Shapley/Scarf model and the definitions of competitive equilibrium and core are presented here in a notation that permits the introduction of lotteries in the next section. The consumption set of trader i is the discrete set $D=\{e_i \mid i=1, 2, ..., n\}$ of n degenerate probability distributions over houses, where e_i is the probability distribution that puts probability 1 on house i. In this notation, trader i's endowment is represented by e_i. Trader i's ordinal ranking over houses may be reinterpreted as a ranking over elements in the set D that has $e_j \succcurlyeq_i e_k$ if and only if house $j \succcurlyeq_i$ house k. An allocation for the economy is an N-tuple $(x_i)_{i \in N}$ of elements in D and is denoted by x.

Definition 1. A *competitive equilibrium* is an allocation x^* and a price vector $p^* \in \Re^n_+$ of the n houses such that (i) for every i, $p^* \cdot x_i^* \leqslant p_i^*$, and $x_i^* \succcurlyeq_i e_j$ for all j such that $p_j^* \leqslant p_i^*$, and (ii) $\sum_{i \in N} x_i^* = \sum_{i \in N} e_i$.

Shapley and Scarf [12] establish the existence of a competitive equilibrium.

Definition 2. An allocation x^* is in the *core* for the Shapley/Scarf model if $\sum_{i \in N} x_i^* = \sum_{i \in N} e_i$, and there is no sub-market $S \subseteq N$ for which there exists an allocation x^S (i.e., an S-tuple $(x_j^S)_{j \in S}$ of elements of D) that has $\sum_{j \in S} x_j^S = \sum_{j \in S} e_j$, and $x_j^S \succ_j x_j^*$ for all $j \in S$.

The core for the Shapley/Scarf model is referred to in this paper as the P-core. Shapley and Scarf [12] establish the non-emptiness of the P-core.

3. Lottery Equilibria

Lotteries are probability distributions over houses. They are introduced to the Shapley/Scarf model by a respecification of the consumption set. From this point on, the consumption set D, defined in Section 2, is extended to include non degenerate lotteries and is given by

$$\Delta^n = \left\{ x_i \in \Re^n_+ \;\middle|\; \sum_{j=1}^{n} x_{ij} = 1 \right\}. \tag{3.1}$$

An allocation of lotteries for the economy is an N-tuple $(x_i)_{i \in N}$ of elements in Δ^n and is denoted by x.

A preference scaling function of trader i's ordinal ranking over houses, $\succcurlyeq_i$, is simply a vector $a_i \in \Re^n$ (i.e., house $j \succcurlyeq_i$ house k if and only if $a_{ij} \geqslant a_{ik}$). Let $\mathscr{A}_i$ denote the set of all preference scaling functions of $\succcurlyeq_i$, and $\mathscr{A} = \mathscr{A}_1 \times \cdots \times \mathscr{A}_n$. Without loss of generality, assume that $\mathscr{A}_i \subseteq \Re^n_{++}$, for all i.

Let $a \in \mathscr{A}$. Assuming traders are both expected utility maximizers and price takers, then given any $p \in \Re^n_+$, trader i solves the problem $(\mathrm{P})_i$:

$$\max_{x_i} \sum_{j=1}^{n} x_{ij} a_{ij} \tag{3.2}$$

$$\text{subject to } \sum_{j=1}^{n} p_j x_{ij} \leqslant p_i \tag{3.3}$$

$$x_i \in \Delta^n. \tag{3.4}$$

For every i, let $\phi_i \colon \Delta^n \to \Re^n_+$, be defined by

$$\phi_i(p) = \{x_i \in \Delta^n : x_i \text{ solves } (\mathrm{P})_i \text{ at } p\}. \tag{3.5}$$

Definition 3. A *lottery equilibrium* is an allocation x^* of lotteries and a price vector $p^* \in \Re^n_+$ such that

(i) $x_i^* \in \phi_i(p^*)$ for $i = 1, 2, ..., n$, and

(ii) $\sum_{i \in N} x_i^* = \sum_{i \in N} e_i$.

Condition (ii) looks very much like a standard market clearing condition. However, caution is needed since the individual demands are lotteries. The reason we are able to define a lottery equilibrium this way is that conditions (i) and (ii) combined ensure the individually demanded lotteries are feasible, in the sense that they can be met by randomizing over allocations of houses. This is demonstrated in Remark 1 below.

Remark 1. By condition (i), $\sum_{j=1}^{n} x_{ij}^* = 1$ for $i = 1, 2, ..., n$. Thus, conditions (i) and (ii) imply that $x^* = (x_{ij}^*)$ forms a doubly stochastic matrix.[1] Redistributions of the houses are permutations $\pi \colon N \to N$, where $\pi(i)$ denotes the house allocated to trader $i \in N$ by the permutation π. Denote the set of permutations by Π. Then, by a direct application of the Birkhoff/von Neumann theorem [14, Lemma 2, p. 46] there exists a *joint lottery* $\ell^* \colon \Pi \to \Re_+$, such that $\ell^* \in \Delta^{n!}$ and

$$x_{ij}^* = \sum_{\pi \in \Pi \colon \pi(i) = j} \ell^*(\pi), \qquad \forall j,\ i \in N. \tag{3.6}$$

Joint lotteries are probability distributions over the set of permutations. The marginal distributions of each joint lottery are thus a feasible allocation of individual lotteries. Remark 1 shows that an N-tuple of individual

[1] Shell and Wright [13] specify sunspot equilibrium allocations using the minimal number of states by constructing a doubly stochastic matrix.

demands that satisfies the conditions of Definition 3 represents the marginal distributions of some joint lottery. Thus, such demands are feasible.

THEOREM 1. *For any $a \in \mathscr{A}$, there exists a lottery equilibrium.*

Proof. The economy may be viewed as a special Arrow–Debreu economy in which each trader has a linear utility function and a compact, convex consumption set, equal to the simplex Δ^n. However, existence of a lottery equilibrium is not immediate since endowments are not contained in the interior of traders' consumption sets. It is necessary to establish the lower semi-continuity of the traders' budget correspondences, and this is done in Appendix A. All other conditions that are required to apply Debreu's [1] fixed point argument are satisfied. ∎

Let LE_a denote the set of lottery equilibrium allocations given $a \in \mathscr{A}$. The set LE_a may contain non-degenerate lottery allocations. However, in any lottery equilibrium, only competitive equilibrium allocations can be realized with positive probability and traders only randomize between houses for which they are indifferent. This is shown in Appendix B. Thus final utilities do not depend on the outcomes of lotteries.

The set LE_a may not contain all competitive equilibrium allocations. In the following example there are two competitive equilibria. One of the competitive equilibrium allocations is a degenerate lottery equilibrium allocation (i.e., is in LE_a) while the other is not.

EXAMPLE 1. Consider an economy with three traders. Suppose the traders' preference scaling functions satisfy

$$a_{11} > a_{13} > a_{12}$$

$$a_{21} > a_{22} = a_{23}$$

$$a_{32} > a_{31} > a_{33}.$$

The allocation $x^* = ((1, 0, 0),\ (0, 0, 1),\ (0, 1, 0))$ is a competitive equilibrium allocation that is supported by prices $p^* \in \Re^3_+$ satisfying $p_1^* > p_2^* = p_3^*$. Furthermore, x^* is a (degenerate) lottery equilibrium allocation at these prices. The allocation $x^{**} = ((1, 0, 0),\ (0, 1, 0),\ (0, 0, 1))$ is another competitive equilibrium allocation that is supported by prices $p^{**} \in \Re^3_+$ satisfying $p_1^{**} > p_2^{**} > p_3^{**}$. However, x^{**} is not a lottery equilibrium allocation. Prices $p' \in \Re^3_+$ that support x^{**} as a lottery equilibrium allocation must also satisfy $p'_1 > p'_2 > p'_3$. Since $a_{21}x_{21} + a_{23}x_{23} > a_{22}$ whenever $x_{21} + x_{23} = 1$ and $x_{21} > 0$, the lottery $x_2 \in \Delta^3$ with $x_{21} + x_{23} = 1$ and $x_{21} = (p'_2 - p'_3)/(p'_1 - p'_3)$ is affordable and preferred to x_2^{**} by trader 2.

The set of lottery equilibrium allocations that do not depend on the specification of traders' preference scaling functions is given by $LE = \bigcap_{a \in \mathscr{A}} LE_a$. If no trader is indifferent between any two of the houses, then LE is non-empty. This is shown in the following proposition.

PROPOSITION 1. *If no trader is indifferent between any two of the houses, then there exists a utility independent lottery equilibrium allocation.*

Proof. Roth and Postlewaite [9] show that there exists a unique competitive equilibrium allocation for the Shapley/Scarf model under the given condition. By Theorem 1 and Appendix B, the unique competitive equilibrium allocation is the only element of LE_a for any $a \in \mathscr{A}$. Therefore LE is non-empty. ▌

4. LOTTERY CORES

For every coalition $S \subseteq N$, the set of S-tuples of lotteries over houses in S is defined by

$$X^S = \left\{ (x_i^S)_{i \in S} \colon x_i^S \in \Delta^n,\ \forall i \in S,\ \sum_{i \in S} x_i^S = \sum_{i \in S} e_i \right\}.$$

As in Remark 1, by the Birkhoff/von Neumann theorem every S-tuple of lotteries in X^S can be shown to be an S-tuple of marginal distributions of some joint lottery over permutations on S. X^S is the set of feasible lottery allocations for the coalition S. When $S = N$, the set X^S is denoted by X.

The notion of core is now applied to the model with traders' preferences represented by $a \in \mathscr{A}$. The resulting core allocations are utility dependent, and are called the L_a-core.

DEFINITION 4. A lottery allocation x^* is in the L_a-core if (i) $x^* \in X$ and (ii) there does not exist a coalition S for which there is an allocation $x^S \in X^S$ such that

$$\sum_{j \in S} a_{ij} x_{ij}^S > \sum_{j \in N} a_{ij} x_{ij}^* \tag{4.1}$$

for all $i \in S$.

The utility independent lottery core (L-core) is then defined from the different L_a-cores as follows.

DEFINITION 5. A lottery allocation x^* is in the L-core if it is in the L_a-core for all $a \in \mathscr{A}$, i.e., L-core $= \bigcap_{a \in \mathscr{A}} \mathrm{L}_a$-core.

The relationship between the P-core and the L-core is now established.

PROPOSITION 2. *P-core* $\subseteq$ *L-core.*

Proof. Choose any $x^* \in$ P-core. By Definition 2, $x^* \in X$. Suppose that $x^* \notin \mathrm{L}_a$-core for some $a \in \mathscr{A}$. Then, there exists a coalition S for which there is an allocation $x^S \in X^S$ such that

$$\sum_{j \in S} a_{ij} x_{ij}^S > \sum_{j \in N} a_{ij} x_{ij}^*, \qquad \forall i \in S. \tag{4.2}$$

Since $x^S \in X^S$, for $i_1 \in S$ there must be $i_2 \in S$ such that

$$a_{i_1 i_2} > \sum_{j \in N} a_{i_1 j} x_{i_1 j}^*. \tag{4.3}$$

Since $x^* \in$ P-core, i_2 cannot be the same as i_1. However, since $i_2 \in S$, by (4.2) there must be an $i_3 \in S$, $i_2 \neq i_3$, such that

$$a_{i_2 i_3} > \sum_{j \in N} a_{i_2 j} x_{i_2 j}^*. \tag{4.4}$$

If $i_3 = i_1$, then by (4.3) and (4.4), the coalition $\{i_1, i_2\}$ can block x^* without using lotteries. But this contradicts the fact that $x^* \in$ P-core. Therefore, $i_3 \notin \{i_1, i_2\}$.

By induction, the above argument can be repeated. Since S is finite, there must exist an integer $m \geqslant 1$ such that

$$a_{i_k i_{k+1}} > \sum_{j \in N} a_{i_k j} x_{i_k j}^*, \qquad k = 1, 2, ..., m, \tag{4.5}$$

and

$$i_{m+1} \in \{i_1, i_2, ..., i_m\}. \tag{4.6}$$

Without loss of generality, assume $i_{m+1} = i_1$, and that m is the least integer that satisfies (4.5) and (4.6). Let $C = \{i_1, i_2, ..., i_m\}$, and let $y^C \subseteq \Delta^n$ be a C-tuple of degenerate lotteries such that $y_{i_k i_{k+1}} = 1$ for $k = 1, 2, ..., m$. Then, coalition C can block x^* via y^C. But this again contradicts the fact that $x^* \in$ P-core. We thus conclude that $x^* \in \mathrm{L}_a$-core. ∎

Remark 2. From this result we see that no P-core allocations can be blocked by any coalition even if its members are allowed to use lotteries. Furthermore, since the non-emptiness of the P-core is established in Shapley and Scarf [12], the L-core is also non-empty.

PROPOSITION 3. *There exists an economy for which the P-core* $\subset$ *L-core.*

Proof. Consider an economy with four traders. Suppose the traders' ordinal preferences over houses are such that

$$\text{house } 4 \succ_1 \text{house } 3 \succ_1 \text{house } 2 \succ_1 \text{house } 1$$

$$\text{house } 4 \succ_2 \text{house } 3 \succ_2 \text{house } 2 \succ_2 \text{house } 1$$

$$\text{house } 2 \succ_3 \text{house } 1 \succ_3 \text{house } 3 \succ_3 \text{house } 4$$

$$\text{house } 2 \succ_4 \text{house } 1 \succ_4 \text{house } 3 \succ_4 \text{house } 4.$$

The allocations $x = ((0, 0, 1, 0), (0, 0, 0, 1), (1, 0, 0, 0), (0, 1, 0, 0))$ and $y = ((0, 0, 1, 0), (0, 0, 0, 1), (0, 1, 0, 0), (1, 0, 0, 0))$ are both in the P-core. Furthermore, the allocation $x^* = ((0, 0, 1, 0), (0, 0, 0, 1), (\alpha, (1-\alpha), 0, 0), ((1-\alpha), \alpha, 0, 0))$, is in the L-core for any $\alpha \in [0, 1]$. ∎

Remark 3. The example presented in the proof of Proposition 3 is such that for $\alpha \in (0, 1)$ the ex post welfare of traders 3 and 4 depends on the outcome of the lottery.

PROPOSITION 4. *There exists an economy for which co(P-core) $\nsubseteq$ L-core.*

Proof. Consider an economy with four traders. It suffices to show *co*(P-core) $\nsubseteq$ L_a-core for some $a \in \mathscr{A}$. Let $a \in \mathscr{A}$ be such that

$$a_{13} > a_{12} > a_{14} > a_{11}$$

$$a_{24} > a_{21} > a_{23} > a_{22}$$

$$a_{34} > a_{32} > a_{31} > a_{33}$$

$$a_{43} > a_{41} > a_{42} > a_{44}.$$

Consider the two P-core allocations $x = ((0, 0, 1, 0), (0, 1, 0, 0), (0, 0, 0, 1), (1, 0, 0, 0))$ and $y = ((1, 0, 0, 0), (0, 0, 0, 1), (0, 1, 0, 0), (0, 0, 1, 0))$. Next, let $\alpha \in (0, 1)$ and consider the allocation $z = \alpha x + (1-\alpha)\, y \in co(\text{P-core})$. The expected utilities of traders 3 and 4 at z are $a_3 \cdot z_3 = \alpha a_{34} + (1-\alpha)\, a_{32}$ and $a_4 \cdot z_4 = \alpha a_{41} + (1-\alpha)\, a_{43}$, respectively. Since $0 < \alpha < 1$, $a_{34} > a_3 \cdot z_3$ and $a_{43} > a_4 \cdot z_4$. Thus, z is blocked by the coalition $S = \{3, 4\}$ via degenerate lotteries $z^S = ((0, 0, 0, 1), (0, 0, 1, 0))$, and thus it is not in the L_a-core. ∎

Proposition 4 establishes that not every randomization over P-core allocations is in the L-core. If the traders' preference scaling functions are known, then there exist economies for which there are L_a-core allocations that are not simply randomizations over P-core allocations. This is demonstrated in the following proposition.

PROPOSITION 5. *There exists an economy for which the L_a-core $\nsubseteq$ co(P-core).*

Proof. Consider an economy with three traders. Suppose the given preference scaling functions are such that

$$a_{12} > a_{13} > a_{11}$$

$$a_{23} > a_{22} > a_{21}$$

$$a_{32} > a_{33} > a_{31}.$$

It is easily verified that $x = ((0, 1, 0), (0, 0, 1), (1, 0, 0))$ is not in the P-core, and $y = ((1, 0, 0), (0, 0, 1), (0, 1, 0))$ is in the P-core. Now consider the allocation $z \in X$ of lotteries with $0 < z_{12} < (a_{32} - a_{33})/(a_{32} - a_{31})$, $z_{12} = z_{31}$, $z_{11} = z_{32} = (1 - z_{12})$, and $z_{23} = 1$. The expected utility to each trader from z is given by $z_{12}a_{12} + z_{11}a_{11}$, a_{23}, and $z_{31}a_{31} + z_{32}a_{32}$, for traders 1, 2, and 3, respectively. Since trader 2 receives her most preferred house, to show z is in the L_a-core, it suffices to show that none of the coalitions $S = \{1\}$, $S = \{3\}$, or $S = \{1, 3\}$ can block z. The coalition $S = \{1\}$ cannot block z since $a_{11} < z_{12}a_{12} + z_{11}a_{11}$. Also, the coalition $S = \{3\}$, cannot block z since $a_{33} < z_{31}a_{31} + z_{32}a_{32}$. Finally, consider the coalition $S = \{1, 3\}$. For any $z^S \in X^S$, $z^S_{11}a_{11} + z^S_{13}a_{13} > z_{12}a_{12} + z_{11}a_{11}$ and $z^S_{31}a_{31} + z^S_{33}a_{33} > z_{31}a_{31} + z_{32}a_{32}$ imply both $z^S_{13} > z_{12}$ and $z^S_{33} > z_{32}$, which is impossible because then $z^S_{13} + z^S_{33} > z_{12} + z_{32} = 1$. Thus, $z \in \mathrm{L}_a$-core. ∎

We conclude this section by establishing that all utility independent lottery equilibrium allocations are contained in the L-core. For any given $a \in \mathscr{A}$, the inclusion of LE_a in the L_a-core is immediate from standard arguments. Therefore, $\bigcap_{a \in \mathscr{A}} LE_a \subseteq \bigcap_{a \in \mathscr{A}} \mathrm{L}_a$-core. This result is stated in the following proposition

PROPOSITION 6. *LE ⊆ L-core.*

4.1. *Non-emptiness of the Lottery Core with More Complex Preferences*

We conclude our discussion of cores with an example from Shapley and Scarf [12, Section 8] that we use to demonstrate another significant role that lotteries can play. In their example, the P-core is empty. However, we demonstrate that when lotteries are allowed the underlying non-transferable utility (NTU) coalitional game of their example is balanced, and hence, by a fundamental theorem in game theory (see Scarf [10]) the core is not empty.

In Shapley and Scarf's example there are three traders, each of whom is endowed with three houses. Denote trader i's endowment of the three houses by (i, i', i''), $i = 1, 2, 3$. Every trader demands only bundles of exactly three houses. So the relevant bundles are those consisting of three houses. Let B denote the set of all these bundles. Let u_i be a utility function representing trader i's preferences for $i=1, 2, 3$. Shapley and Scarf assume for $i=1, 2, 3$, and for $b \in B$, that

$$u_i(b)=\begin{cases} 2 & \text{if } \ b=(i, \alpha_i, \alpha_i') \\ 1 & \text{if } \ b=(i'', \beta_i', \beta_i'') \\ 0 & \text{otherwise,} \end{cases}$$

where $(\alpha_1, \beta_1)=(2, 3)$, $(\alpha_2, \beta_2)=(3, 1)$, and $(\alpha_3, \beta_3)=(1, 2)$. When lotteries are not allowed, the characteristic function V of the underlying NTU coalitional game is determined by

$$V(\{i\})=\{u \in \mathfrak{R}^3\colon u_i \leqslant 0\}, \qquad i \in \{1, 2, 3\};$$

$$V(\{i, j\})=\{u \in \mathfrak{R}^3\colon u_i \leqslant 2, u_j \leqslant 1\}, \qquad i, j \in \{1, 2, 3\} \text{ with } i<j; \text{ and}$$

$$V(\{1, 2, 3\})=\{(2, 1, 0), (1, 0, 2), (0, 2, 1)\}-\mathfrak{R}^3_+.$$

Since the utility vector $(1, 1, 1) \in \bigcap_{i<j} V(\{i, j\}) \setminus V(\{1, 2, 3\})$, the game is not balanced, and furthermore, the P-core is empty (see Shapley and Scarf [12]). Since $V(\{i\})$ and $V(\{i, j\})$ all have single "corners," introducing lotteries does not change these utility sets. Note also that any utility vector $(u_1, u_2, u_3) \in \bigcap_{i<j} V(\{i, j\})$ is dominated by the utility vector $(1, 1, 1)$. Consider the joint lottery ℓ over the set of allocations: $\ell(b_1, b_2', b_3'')=\ell(b_1', b_2'', b_3)=\ell(b_1'', b_2 b_3')=1/3$, where $b_i, b_i', b_i'' \in B$ are such that $u_i(b_i)=2$, $u_i(b_i')=1$, and $b_i''=(i, i', i'')$ for $i=1, 2, 3$. The joint lottery ℓ is clearly well defined and it induces a feasible allocation (x_1, x_2, x_3) of individual lotteries with $x_i(b_i)=x_i(b_i')=x_i(b_i'')=1/3$, $i=1, 2, 3$. Furthermore, $x_i(b_i)\, u_i(b_i)+x_i(b_i')\, u_i(b_i')+x_i(b_i'')\, u_i(b_i'')=1$, $i=1, 2, 3$, and so the resulting NTU coalitional game is balanced when lotteries are allowed.

5. Concluding Remarks

In the late 1980's the notion of the core of NTU coalitional games was generalized to allow for the possibility of a mediator who proposes blocking strategies that involve random blocking coalitions (see Myerson [6]). This generalized notion of the core has a close relationship to the concept formalized by Shapley [11] known as the inner core (see Qin [7]). This paper examines core theory when random allocations are permitted. By allowing random allocations the set of possible blocking allocations and the set of possible core allocations are increased.

Shapley and Scarf [12, Section 8] provide an example of an exchange model with more complex preferences for which the P-core is empty. For their example, it is shown here that if randomization is permitted, the resulting NTU coalitional game is balanced and the lottery core is not empty. Whether the lottery core is always non-empty in the model with more complex preferences is an open question.

The core concept of this paper is related to one developed independently for sunspot economies by Goenka and Shell [3]. They consider the core for sunspot economies when blocking proposals may include a respecification of the extrinsic probability space.

Appendix A.

The budget correspondence ψ_i is lower semi-continuous.

Define $\psi_i\colon \Delta^n \to \Delta^n$ by $\psi_i(p) = \{x_i \in \Delta^n \mid p \cdot x_i \leqslant p_i\}$. Let $\bar{p}$ be any element in Δ^n and let (p^k) be any sequence that converges to $\bar{p}$. Choose any $\bar{x}_i \in \psi_i(\bar{p})$.

Case 1. $\bar{p} \cdot \bar{x}_i < \bar{p}_i$.

Since $p^k \to \bar{p}$, there exists k_0 such that $p^k \cdot \bar{x}_i < p_i^k$ for all $k \geqslant k_0$. For any k, let x_i^k be any element in $\psi_i(p^k)$ if $k < k_0$; and let $x_i^k = \bar{x}_i$ if $k \geqslant k_0$. Then, $x_i^k \in \psi_i(p^k)$ for all k and $x_i^k \to \bar{x}_i$.

Case 2. $\bar{p} \cdot \bar{x}_i = \bar{p}_i$ and $\bar{p} \cdot \bar{x}_i > \min\{\bar{p} \cdot x_i \mid x_i \in \Delta^n\}$.

Let $\bar{y}_i$ be an element in Δ^n such that $\bar{p} \cdot \bar{y}_i = \min\{\bar{p} \cdot x_i \mid x_i \in \Delta^n\}$. Then, there exists k_0 such that for any $k \geqslant k_0$, $p^k \cdot (\alpha \bar{x}_i + (1-\alpha)\, \bar{y}_i) \leqslant p_i^k$ holds for some $\alpha \in [0, 1]$. For $k \geqslant k_0$, let $\alpha_k = \max\{\alpha \in [0, 1] \mid p^k \cdot (\alpha \bar{x}_i + (1-\alpha)\, \bar{y}_i) \leqslant p_i^k\}$. Then,

$$p^k \cdot (\alpha_k \bar{x}_i + (1-\alpha_k)\, \bar{y}_i) \leqslant p_i^k, \qquad k \geqslant k_0. \tag{6.1}$$

Thus, $x^k = \alpha_k \bar{x}_i + (1-\alpha_k)\, \bar{y}_i \in \psi_i(p^k)$ for all $k \geqslant k_0$. For any $k < k_0$, let α_k be any element in [0,1]. Suppose (α_k) does not converge to 1. Then, there exists a subsequence (α_{k_m}) of (α_k) such that

$$\alpha_{k_m} < \varepsilon, \qquad m = 1, 2, \ldots \tag{6.2}$$

for some $0 < \varepsilon < 1$. By construction,

$$p^{k_m} \cdot (\varepsilon \bar{x}_i + (1-\varepsilon)\, \bar{y}_i) > p_i^{k_m}, \qquad m = 1, 2, \ldots \tag{6.3}$$

Letting $m \to \infty$ on both sides of (6.3),

$$\bar{p} \cdot (\varepsilon \bar{x}_i + (1-\varepsilon)\, \bar{y}_i) \geqslant \bar{p}_i, \tag{6.4}$$

which implies that $\bar{p} \cdot \bar{x}_i > \bar{p}_i$, because $\varepsilon > 0$ and $\bar{p} \cdot \bar{y}_i < \bar{p} \cdot \bar{x}_i$. Thus, we have a contradiction. Let x_i^k be defined as above for $k \geqslant k_0$, and let x^k be any element in $\psi_i(p^k)$ for $k<k_0$. Then, $x_i^k \in \psi_i(p^k)$ for all k and $x_i^k \to \bar{x}_i$ as $k \to \infty$.

Case 3. $\bar{p}\cdot\bar{x}_i=\bar{p}_i$ and $\bar{p}\cdot\bar{x}_i=\min\{\bar{p}\cdot x_i \mid x_i \in \Delta^n\}$.

For any $p \in \Delta^n$, let $J(p)=\{j \mid p_j=\min_{j'} p_{j'}\}$. Let $J=J(\bar{p})$, and $\bar{p}\cdot\bar{x}_i=\bar{p}^J\cdot\bar{x}_i^J+\bar{p}^{N\backslash J}\cdot\bar{x}_i^{N\backslash J}$. Then $\bar{p}\cdot\bar{x}_i=\min\{\bar{p}\cdot x_i \mid x_i \in \Delta^n\}$ implies $\bar{x}_{ij}=0$ for $j \in N\backslash J$, and hence $\bar{p}\cdot\bar{x}_i=\bar{p}_i$ implies $i \in J$. This shows that $N\backslash J(\bar{x}_i) \subseteq J$. Since $p^k \to \bar{p}$, there exists k_0 such that $J(p^k)=J$ for all $k \geqslant k_0$. Thus, since $\bar{x}_i \in \Delta^n$, and both $N\backslash J(\bar{x}_i) \subseteq J(p^k)$ and $i \in J(p^k)$ hold for all $k \geqslant k_0$, we have

$$p^k\cdot\bar{x}_i=p_i^k, \qquad k \geqslant k_0. \tag{6.5}$$

Let x_i^k be any element in $\psi_i(p^k)$ if $k<k_0$; and let $x_i^k=\bar{x}_i$ if $k \geqslant k_0$. Then, by (6.5), $x_i^k \in \psi_i(p^k)$ for all k and $x_i^k \to \bar{x}_i$ as $k \to \infty$.

To summarize, we have proven that there exists a sequence (x_i^k) in Δ^n such that $x_i^k \in \psi_i(p^k)$ for all k and $x_i^k \to \bar{x}_i$ as $k \to \infty$. Therefore, by Theorem 2 of Hildenbrand [4, p. 27], ψ_i is lower semi-continuous.

Appendix B.

Given $a \in \mathscr{A}$, for any lottery equilibrium there exists a partition of N into subsets S^k, $k=1, ..., m$, such that all of the houses owned by the members of the subset S^k are priced the same, the demands of traders in any subset S^k form a doubly-stochastic matrix, and only competitive equilibrium allocations can be realized with positive probability.

For any $\pi \in \Pi$, two traders i and j are said to be connected by π if $i=j$ or there is some integer $m \geqslant 1$ and a sequence $\{i_0, i_1, ..., i_m\}$ of traders such that $i_0=i$, $i_m=j$, $\pi(i_{k-1})=i_k$ for $k=1, 2, ..., m$, and $\pi(i_m)=i_0$. Let $P(\pi)$ denote a partition of N such that every coalition in $P(\pi)$ is connected and is not contained in any other connected coalition.

Let $((x_i^*)_{i\in N}, p^*)$ be a lottery equilibrium and let $\ell^* \in \Delta^{n!}$ be a joint lottery so that for any i and any j

$$x_{ij}^*=\sum_{\pi:\,\pi(i)=j} \ell^*(\pi). \tag{6.6}$$

Set $S^1=\{j \in N\colon p_j^*=\min_{i\in N} p_i^*\}$. If $S^1 \neq N$, set $S^2=\{j \in N\backslash S^1\colon p_j^*=\min_{i\in N\backslash S^1} p_i^*\}$. Suppose $S^1, ..., S^k$ have been defined. If $S^1 \cup \cdots \cup S^k \neq N$, set $S^{k+1}=\{j \in N\backslash\bigcup_{t=1}^k S^t\colon p_j^*=\min_{i\in N\backslash\bigcup_{t=1}^k S^t} p_i^*\}$. Since $n<\infty$, there exists some integer $m \geqslant 1$ such that $S^1 \cup \cdots \cup S^k=N$. Then, for any $i \in S^1$, it follows from trader i's budget constraint that $x_{ij}=0$ for any $x_i \in \phi_i(x_i)$ and $j \notin S^1$. That is,

$$\sum_{h\in S^1} x_{ih}^*=\sum_{h\in S^1} x_{hj}^*=1 \tag{6.7}$$

holds for any $i, j \in S^1$. Note that (6.7) together with condition (ii) of Definition 3 implies $x^*_{ij} = 0$ for any $i \notin S^1$ and $j \in S^1$. Using a recursive argument, it follows that for $1 < k \leqslant m$,

$$\sum_{h \in S^k} x^*_{ih} = \sum_{h \in S^k} x^*_{hj} = 1 \tag{6.8}$$

holds for any $i, j \in S^k$.

Since, for any $\pi \in \Pi$ with $\ell^*(\pi) > 0$ the partition $P(\pi)$ is no coarser than the partition $(S^1, ..., S^m)$, $P(\pi)$ consists of only *top trading cycles* (see Shapley and Scarf [12]). Therefore, any $\pi \in \Pi$ with $\ell^*(\pi) > 0$ corresponds to a competitive equilibrium allocation $(y^*_i)_{i \in N}$ with $y^*_{i\pi(i)} = 1$ for all $i \in N$.

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