Pergamon

BRAESS' PARADOX: SOME NEW INSIGHTS

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Abstract—This paper examines some properties of the well-known Braess' paradox of traffic flow, in the context of the classical network configuration used by Braess. The paper shows that whether Braess' paradox does or does not occur depends on the conditions of the problem; namely, the link congestion function parameters and the demand for travel. In particular, this paper shows that for a given network with a given set of link congestion functions, Braess' paradox occurs only if the total demand for travel falls within a certain intermediate range of values (the bounds of which are dependent on the link congestion function parameters). The paper also shows that, depending on the problem parameters, adding a new link might not lead to a reduction in total system travel time, even if users are charged the marginal cost of traveling. On the other hand, there are ranges of values for the problem parameters for which the new link reduces total system travel time, as long as marginal cost pricing is implemented. © 1997 Elsevier Science Ltd

1. INTRODUCTION AND BACKGROUND

In 1968, an operations researcher named Dietrich Braess published a paper in which he showed that adding a new link to a transportation network might not improve the operation of the system, in the sense of reducing the total vehicle-minutes of travel in the system (Braess, 1968 as described by Murchland, 1970). Of course, this is a seemingly counter-intuitive and puzzling result, and the phenomenon soon became known as Braess' paradox of traffic flow. This phenomenon is well known in the transportation field and has been the subject of considerable discussion in both the transportation and more general scientific literature (Arnott and Small, 1994; Bass, 1992; Calvert and Keady, 1993; Cohen and Jeffries, 1994; Cohen and Kelly, 1990; Dafermos and Nagurney, 1984; Fisk, 1979; Lam, 1988; Murchland, 1970; Steinberg and Zangwill, 1983).

Braess' paradox occurs because, under the common approach to pricing road usage, users attempt to minimize their own travel time while ignoring the effect of their decisions on other travelers. (Some other, lesser known paradoxes of traffic flow occur for the same reason, as discussed in a recent paper by Arnott and Small, 1994[1]). The common pricing method, namely average cost pricing, leads to user equilibrium assignment in which each user minimizes his/her own travel time between an origin and a destination. As a result, it is possible for the total system travel time to increase following an expansion of the transportation network, since even if some travelers are better off using the new link, they can contribute to increasing congestion for other travelers. Conversely, Braess' paradox disappears if road usage is priced at marginal cost; under such pricing the system optimal flow pattern is achieved and total system travel time is minimized. Since we now know why adding a new link might cause the total system travel time to increase, the phenomenon known as Braess' paradox is not really paradoxical, and Sheffi (1985) refers to this and similar traffic flow phenomena as 'pseudo-paradoxes'. However, because the phenomenon discussed in this paper is still commonly referred to as Braess' paradox, we refer to it by that name here.

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[1] Our interest in the subject of Braess' paradox arose when the first author asked the students in the Transportation Systems Analysis class at Duke University to read the Arnott and Small (1994) paper and solve some problems based on the examples in that paper. One of the assigned problems asked the students to show that, by pricing the use of the road network at the marginal cost, Braess' paradox is resolved for the example problem in the Arnott and Small paper. The students found that while the paradox is resolved through marginal cost pricing, the new link is not used in the system optimal solution, and therefore the network expansion would clearly not be warranted in that example. This finding motivated us to examine the conditions under which the system optimal solution leads to the new link carrying some positive amount of flow (thus improving the performance of the system). Subsequently, we decided to also examine the conditions under which the paradox itself occurs.

It is clear that under a system optimal assignment, adding a new link cannot leave the transportation system (and its users) worse off; the new link can always be assigned zero flow in this case, and the total system travel time would be the same as that without the new link. Of course, if it is best for the new link to carry no traffic, then it makes no sense whatsoever to add the link to the network. On the other hand, if expanding the system reduces the total system travel time under marginal cost pricing, then the expansion of the network might be justified, depending on the overall set of costs and benefits associated with the new link.

This paper examines some properties of Braess' paradox of traffic flow, in the context of the classical network configuration used by Braess and most researchers who have examined the problem he first described. The paper identifies the conditions of the problem (the link congestion function parameters and the total demand for travel) that determine whether the paradox does or does not occur in the classical network. The paper also identifies those situations where the new link improves the traffic flow (under marginal cost pricing), and those situations where the link makes no difference to the operation of the system (even under marginal cost pricing). Specifically, four mutually exclusive and collectively exhaustive cases are identified and discussed in this paper, namely:

1. Braess' paradox does not occur because the demand is too low. In this case, network expansion results in reduced total system travel time under the conventional average cost pricing approach.
2. Braess' paradox does not occur because the demand is too high. Again, network expansion results in reduced total system travel time under the conventional average cost pricing approach.
3. Braess' paradox occurs, but network expansion leads to improved system performance if marginal cost pricing is applied.
4. Braess' paradox occurs, and network expansion leaves the total system travel time unchanged under marginal cost pricing (because the new link does not carry any traffic in the optimal flow pattern).

The remainder of this paper is organized as follows. In Section 2 we formulate the problem and show, using the configuration of the classical network for Braess' paradox, that the paradox occurs only for certain values of the problem parameters (link congestion function coefficients and total travel demand). In Section 3 we show, again in the context of the classical network for Braess' paradox, that adding a new link and subjecting travelers to marginal cost pricing can either improve overall system performance or leave it unaffected, depending upon the values of the problem parameters. Thus, while marginal cost pricing always prevents Braess' paradox from occurring, we find that it does not necessarily ensure that the additional link will improve the overall system performance. The analytical results in Sections 2 and 3 are illustrated using the numerical examples presented by Braess (1968) and Arnott and Small (1994). In the final section, Section 4, we state our conclusions.

2. WHEN DOES BRAESS' PARADOX OCCUR?

In this section of the paper we examine the conditions under which Braess' paradox occurs in the classical network employed for illustrating this phenomenon. We first describe the network.

2.1. Network description

Consider the simple transportation network shown in Fig. 1, with a single origin node o, and a single destination node r. This network has the same configuration as that used by Braess and most of the researchers who subsequently addressed the problem he first identified. Figure 1(a) depicts the network before addition of the new link pq, while Fig. 1(b) depicts the network after addition of this link. Figure 1(c) depicts the two routes (opr and oqr) from o to r for the four-link network shown in Fig. 1(a), while Fig. 1(d) shows the additional route ($opqr$) from o to r that is made possible by the addition of the link pq. Note that the links op and qr are assumed to be bottleneck-type situations in which the travel time increases rapidly as a function of the flow on the link.

Let the total demand for travel from origin o to destination r be Q, and let this demand be fixed (i.e. inelastic), as in the problem formulated by Braess. Further, assume that the link volume–travel time functions are linear, as in the problem formulated by Braess. Specifically, let the link travel time functions be given by:

$$t_{ij} = \alpha_{ij} + \beta_{ij} f_{ij}, \tag{1}$$
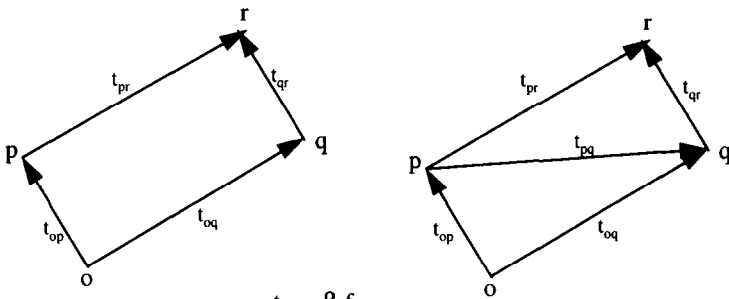
where

t_{ij}	is the travel time on link ij,
α_{ij}	is the free flow travel time on link ij,
β_{ij}	is the delay parameter for link ij (the increase in travel time per unit increase in the flow on link ij), and
f_{ij}	is the flow on link ij.

As in the classical problem for Braess' paradox, we assume that the problem is symmetric, so that:

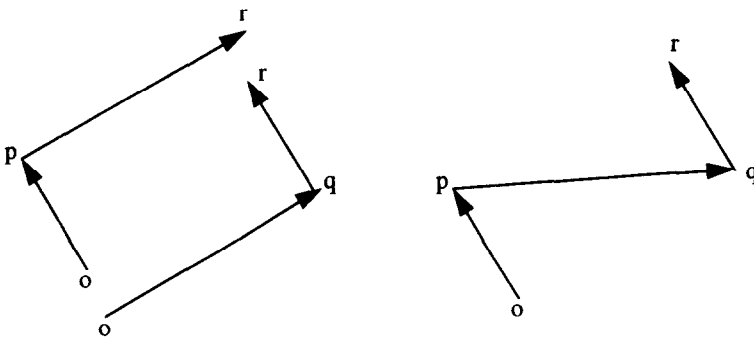$$\alpha_{op} = \alpha_{qr}, \quad \alpha_{oq} = \alpha_{pr}$$

and

$$\beta_{op} = \beta_{qr}, \quad \beta_{oq} = \beta_{pr}.$$

$$t_{op} = \beta_1 f_{op}$$
$$t_{qr} = \beta_1 f_{qr}$$
$$t_{oq} = \alpha_1 + \beta_2 f_{oq}$$
$$t_{pr} = \alpha_1 + \beta_2 f_{pr}$$
$$t_{pq} = \alpha_2 + \beta_2 f_{pq}$$

(a) Four-link network   (b) Five-link network

(c) Two possible routes in the   (d) Additional route created by the
    four-link network   new link pq

Fig. 1. Description of the network and identification of alternate routes.

Further, the bottleneck-type links (*op* and *qr*) are assumed to be very short and hence to have zero free flow travel time. Thus, we have:

$$\alpha_{op} = \alpha_{qr} = 0.$$

Finally, the delay parameter for the new link (*pq*) is assumed to be the same as for links *oq* and *qr*. Therefore,

$$\beta_{pq} = \beta_{oq} = \beta_{pr}.$$

Now, let

$$\alpha_{oq} = \alpha_{pr} = \alpha_1,$$

$$\alpha_{pq} = \alpha_2,$$

$$\beta_{op} = \beta_{qr} = \beta_1,$$

and

$$\beta_{pq} = \beta_{oq} = \beta_{pr} = \beta_2.$$

Therefore, the link congestion functions (eqn 1) become:

$$t_{op} = \beta_1 f_{op}, \tag{2a}$$

$$t_{qr} = \beta_1 f_{qr}, \tag{2b}$$

$$t_{oq} = \alpha_1 + \beta_2 f_{oq}, \tag{2c}$$

$$t_{pr} = \alpha_1 + \beta_2 f_{pr}, \tag{2d}$$

$$t_{pq} = \alpha_2 + \beta_2 f_{pq}. \tag{2e}$$

Now, defining

t^k as the travel time from *o* to *r* along route *k* and
f^k as the flow from *o* to *r* along route *k*, we have

$$t^1 = t_{op} + t_{pr},$$

$$t^2 = t_{oq} + t_{qr},$$

$$t^3 = t_{op} + t_{pq} + t_{qr},$$

and

$$Q = f^1 + f^2 + f^3.$$

2.2. User equilibrium flow pattern: four-link network

The user equilibrium solution for the four-link network flow problem can be solved by inspection because of the symmetry of this problem. In this case:

$$f^1 = f^2 = Q/2$$

and

$$t^1 = t^2 = \frac{Q(\beta_1 + B_2)}{2} + \alpha_1.$$

Hence the total system travel time for the user equilibrium solution (before addition of the new link) is given by:

$$T^4 = \frac{Q}{2}(t^1) + \frac{Q}{2}(t^2) = \frac{Q^2(\beta_1 + \beta_2)}{2} + Q\alpha_1, \tag{3}$$

where

T^4 denotes the total system travel time for the average cost solution (user equilibrium) for the four-link network.

2.3. User equilibrium flow pattern: five-link network

Now, we examine what happens when the link pq is added to the four-link network, as shown in Fig. 1(b). With the addition of this link, there are three possible routes from o to r (opr, oqr, and opqr—see Fig. 1(c) and 1(d)), and because the network has been expanded, one would naturally expect that the total system travel time would not get worse. In fact, one would in general expect the total system travel time to decrease if an existing network were expanded. However, as Braess showed, one's intuition is not necessarily correct in this case. Braess showed that the addition of link pq increases the total system travel time in the case where the problem is characterized by the parameters shown in the second column of Table 1. Specifically, in this case, if the flows are distributed according to the user equilibrium criterion, the total system travel time increases from 498 vehicle-minutes to 552 vehicle-minutes when the network is expanded.

A relevant question to ask at this point is whether the same situation arises for all possible values of the problem parameters. The short answer to this question is "no". To examine this question, we first note that the general expressions for the solution to the user equilibrium problem after the addition of the fifth link can be obtained by equating the travel time on each of the three possible routes from o to r, i.e. by setting

$$t^1 = t^2 = t^3.$$

In this case, the user equilibrium solution is given by:

$$f^1 = f^2 = \frac{\alpha_2 - \alpha_1 + Q(\beta_1 + \beta_2)}{\beta_1 + 3\beta_2}, \tag{4a}$$

$$f^3 = Q - 2f^1 \tag{4b}$$

and

$$t^1 = t^2 = t^3 = \alpha_1 + Q\beta_1 + (\beta_2 - \beta_1)\left[\frac{\alpha_2 - \alpha_1 + Q(\beta_1 + \beta_2)}{\beta_1 + 3\beta_2}\right]. \tag{4c}$$

Equations (4a–c) apply only if the flow on all three routes is positive. However, for some values of the problem parameters, route 3 carries no traffic, while for other values of the parameters, routes 1 and 2 carry no traffic. From eqns (4a,b) we find that if

$$Q \le \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}, \tag{5}$$

then routes 1 and 2 carry no traffic, and all the traffic uses route 3. That is, if the inequality above is satisfied, then:

$$f^1 = f^2 = 0 \tag{6a}$$

and

$$f^3 = Q, \tag{6b}$$

Table 1. Parameters characterizing Braess' classical example and Arnott and Small's problem

Parameter	Braess' example (Braess, 1968)	Arnott and Small's example (Arnott and Small, 1994)
α_1	50	15
α_2	10	7.5
β_1	10	0.01
β_2	1	0
Q	6	1000

Note: α_1, α_2, β_1, β_2 are parameters characterizing the link congestion functions on the various links of the transportation network shown in Fig. 1 (see eqns 2a–e), and Q is the total demand for travel from origin o to destination r).

in which case

$$t^1 = t^2 = 0 \tag{6c}$$

and

$$t^3 = Q(2\beta_1 + \beta_2) + \alpha_2. \tag{6d}$$

Further, from eqns (4a,b) we find that if:

$$Q \geq \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}, \tag{7}$$

then route 3 carries no traffic, and all the traffic uses routes 1 and 2. That is, if eqn (7) is satisfied, then:

$$f^1 = f^2 = Q/2 \tag{8a}$$

and

$$f^3 = 0, \tag{8b}$$

in which case

$$t^1 = t^2 = \frac{Q(\beta_1 + \beta_2)}{2} + \alpha_1 \tag{8c}$$

and

$$t^3 = 0. \tag{8d}$$

We note that we wrote eqns (5) and (7) above for Q in terms of the other parameters, but we could alternatively have written an expression for one of the other parameters. We note also that eqns (4a–c) above, which apply to the case where there is flow along all three routes, are relevant for Q in the range:

$$\frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} < \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}.$$

Figures 2(a) and 3(a) illustrate how the flows along the three routes vary as a function of Q, in the average cost pricing case, for the parameters of the problems described by Braess and Arnott and Small, respectively (see Table 1 for the problem parameters).
Now, in general,

$$T^5 = f^1 t^1 + f^2 t^2 + f^3 t^3, \tag{9}$$

where

T^5 denotes the total system travel time for the average cost solution (user equilibrium) for the five-link network (i.e. after the addition of the link pq). There are actually three distinct expressions for T^5, corresponding to each of the three cases described above. If the flow on the original routes (1 and 2) is zero, then $T^5 = f^3 t^3$, with f^3 and t^3 given by eqn (6), while if the flow on the new route is zero, then $T^5 = f^1 t^1 + f^2 t^2$, with f^1, f^2, t^1 and t^2 given by eqn (8). Finally, if all three routes carry flow, then T^5 is given by eqn (9), with the flows and travel times given by eqn (4). Thus, T^5 is given by:

$$T^5 = Q^2(2\beta_1 + \beta_2) + Q\alpha_2 \quad \text{if} \, Q \leq \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}, \tag{10a}$$

$$= Q\left[\alpha_1 + Q\beta_1 + (\beta_2 - \beta_1)\left(\frac{\alpha_2 - \alpha_1 + Q(\beta_1 + \beta_2)}{\beta_1 + 3\beta_2}\right)\right] \quad \text{if} \, \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} < \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}, \tag{10b}$$

$$= \frac{Q^2(\beta_1 + \beta_2)}{2} + Q\alpha_1 \quad \text{if} \, Q \geq \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}. \tag{10c}$$
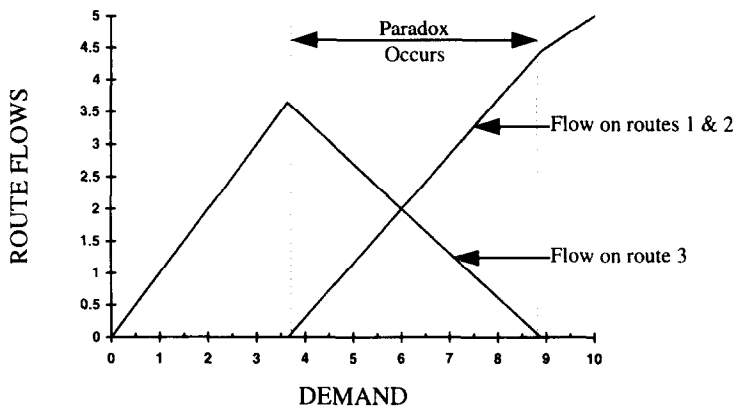
To determine under what circumstances the addition of the new link worsens the overall system performance, we simply need to find the values of the five parameters of the problem (α_1, α_2, β_1, β_2, Q) for which the total travel time in the five-link network (eqn 10) exceeds the total travel time of the original four-link network (eqn 3). After some manipulation and simplification we find that:

$$T^5 > T^4 \quad \text{iff} \, Q(\beta_1 - \beta_2) < 2(\alpha_1 - \alpha_2) \text{or} Q(3\beta_1 + \beta_2) > 2(\alpha_1 - \alpha_2). \tag{11}$$

Solving eqn (11) for Q yields:

$$\frac{2(\alpha_1 - \alpha_2)}{3\beta_1 + \beta_2} < \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}. \tag{12}$$

Equation (12) above yields the interesting observation that, for a given set of parameters for the link congestion functions in the classical Braess' network, the paradoxical situation occurs only if the flow is within the range given by this equation. We note that the upper limit of this range is finite unless $\beta_1 = \beta_2$ (in which case the upper limit on Q for the occurrence of Braess' paradox is). On the other hand, if $\alpha_1 = \alpha_2$, then Braess' paradox will not occur for any positive value of Q. The result given in eqn (12) is somewhat surprising. In essence, it says that, for a given set of link congestion functions, Braess' paradox will not occur if the demand for travel is 'too large' or 'too small'. The former case implies that a network, subject to Braess' paradox under a given level of demand, might 'grow out' of this condition when the travel demand increases over time. More
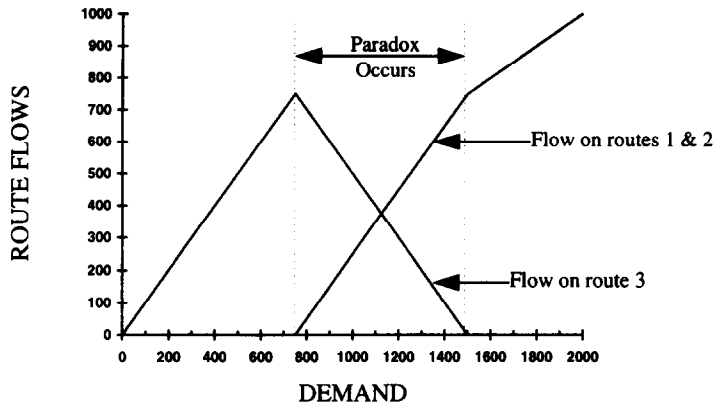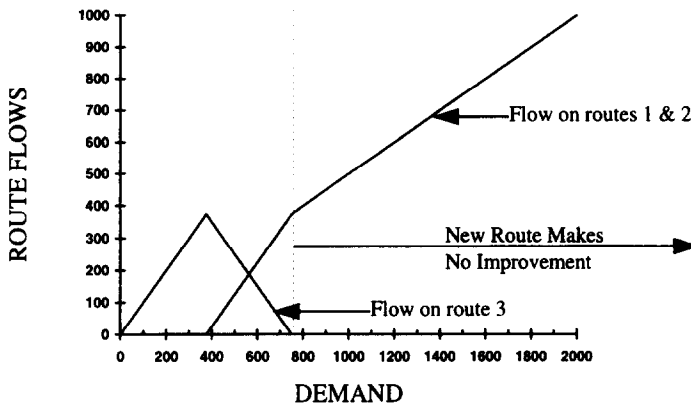

(a) Average Cost Pricing

(b) Marginal Cost Pricing

Fig. 2. Route flows as a function of demand — Braess' example.

(a) Average Cost Pricing

(b) Marginal Cost Pricing

Fig. 3. Route flows as a function of demand — Arnott and Small's example.

importantly, the latter case implies that a network, in which the paradox does not occur at a given (low) level of demand, might 'grow into' the paradox as the demand increases over time[2].

For example, if we set the parameters of the link congestion functions at the values given in Table 1 for Braess' problem, we find that the paradox occurs if:

$$2.58 < Q < 8.89,$$

while for the values used in the Arnott and Small example, we find that the paradox occurs if:

$$500 < Q < 1500.$$

Note that Braess used a demand of 6 units in his example, while Arnott and Small used a demand of 1000 units in their example, resulting in the paradoxical situation in both cases. We have shown here that demands outside the above ranges would not result in the paradoxical outcome.

3. UNDER WHAT CIRCUMSTANCES DOES THE NEW LINK IMPROVE OVERALL SYSTEM PERFORMANCE UNDER MARGINAL COST PRICING?

We have seen above that, for the network shown in Fig. 1, the addition of link pq can lead to increased total system travel time, if the flow from o to r is distributed according to the user

equilibrium assignment rule. It is now well known that this counter-intuitive result will not occur if users are distributed across the different routes according to the system optimal assignment rule. This latter distribution is obtained by charging users the marginal cost of traveling.

The question addressed in this section of the paper is "When does adding the new link to the existing network reduce the total system travel time if users are charged the marginal cost of traveling?" To answer this question, we first derive expressions for the route travel times in the case of marginal cost pricing. The link travel time–volume relationships given in eqns (2a–e) are average cost equations. The marginal cost equations can be derived by multiplying each average cost equation by the flow (f) and then differentiating the resultant equation with respect to flow. For the classical Braess' network, the route marginal cost equations turn out to be:

$$t^{1*} = \alpha_1 + 2\beta_1 f_{op} + 2\beta_2 f_{pr},$$

$$t^{2*} = \alpha_1 + 2\beta_1 f_{qr} + 2\beta_2 f_{oq},$$

$$t^{3*} = \alpha_2 + 2\beta_1 f_{op} + 2\beta_2 f_{pq} + 2\beta_1 f_{qr},$$

where

t^{k*} is the marginal travel time from o to r along route k.

Using the above marginal cost equations, we can easily find the system optimal solution to the five-link network shown in Fig. 1(b). This is accomplished by equating the marginal travel time on each of the three possible routes from o to r. The solution to this problem turns out to be:

$$f^{1*} = f^{2*} = \frac{\alpha_2 - \alpha_2 + 2Q(\beta_1 + \beta_2)}{2(\beta_1 + 3\beta_2)} \qquad (13)$$

$$f^{3*} = Q - 2f^{1*}. \qquad (14)$$

Now, in networks subject to Braess' paradox, marginal cost pricing will lead to an improvement in system performance only if the new link carries some traffic. That is, marginal cost pricing will lead to improved system performance if:

$$f^{3*} > 0.$$

From eqns (13) and (14), we can easily determine that the flow on the new link is positive, in the marginal cost pricing situation, if:

$$Q < \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}. \qquad (15)$$

That is, if Q satisfies the above inequality, then the new link carries flow, and the total system travel time is reduced by the addition of the link pq, as long as users are charged the marginal cost of traveling. On the other hand, if Q equals or exceeds the bound given in eqn (15) then the new link carries no traffic and cannot be justified even under marginal cost pricing.

Thus, we find that adding a new link to a transportation system may not lead to a reduction in total system travel time, even when users are charged the marginal cost of travel. Whether or not this situation occurs for the network represented in Fig. 1, depends on the parameters of the problem (the link travel time–volume relationships and the demand for travel). Of course, if adding the new link does not decrease total system travel time, the link serves no purpose and should not be built. On the other hand, if total system travel time is reduced, the new link may be warranted, but this depends on an assessment of all the relevant costs and benefits.

For the link congestion function parameters of the example problems employed by Braess and Arnott and Small (see Table 1), we find that the upper bounds on Q for which the new link carries a positive traffic flow, are as follows:

$Q < 4.44$ (for Braess' example)

and $Q < 750$ (for Arnott and Small's example).

Fig. 2(b) and Fig. 3(b) illustrate how the flows along the three routes vary as a function of Q, in the marginal cost pricing case, for the parameters of the problems described by Braess and Arnott and Small, respectively.

Now, in order to determine whether marginal cost pricing does or does not lead to improved system performance through network expansion in circumstances when Braess' paradox occurs, we need to examine the relationships between the bounds in eqns (12) and (15). It can easily be seen from these equations that the (upper) bound on Q in eqn 15 is exactly one-half of the upper bound on Q in eqn (12) in this case (the symmetrical network conventionally used to illustrate Braess' paradox). It can also easily be shown that the lower bound on Q in eqn (12) is less than the (upper) bound on Q in eqn (15). Therefore, the bound on Q in eqn (15) lies between the bounds on Q in eqn (12). In other words, for a given network, the range of demand in which Braess' paradox occurs can be divided into two sub-ranges. In one of these sub-ranges, namely, when

$$\frac{2(\alpha_1 - \alpha_2)}{2\beta_1 + \beta_2} < \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}, \tag{16}$$

marginal cost pricing results in a flow pattern in which the additional link is used, and the overall system performance is improved. On the other hand, in the sub-range

$$\frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2} < \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}, \tag{17}$$

marginal cost pricing results in a flow pattern in which the additional link is not used. In this case, the additional link is not warranted, even under marginal cost pricing.

Therefore, for a given network, there are four situations that may occur, depending on the demand for travel, as follows.

1. Braess' paradox does not occur because the demand is too low:

$$Q \leq \frac{2(\alpha_1 - \alpha_2)}{3\beta_1 + \beta_2}.$$

2. Braess' paradox occurs, but network expansion leads to improved system performance under marginal cost pricing:

$$\frac{2(\alpha_1 - \alpha_2)}{3\beta_1 + \beta_2} < \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}.$$

3. Braess' paradox occurs, and network expansion does not lead to improved system performance even under marginal cost pricing:

$$\frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2} < \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}.$$

4. Braess' paradox does not occur because the demand is too high:

$$Q \geq \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}.$$

The analytical and numerical results reported above are summarized in Table 2.

We also note that our work sheds some light on the results reported in Steinberg and Zangwill's (1983) paper. In deriving the conditions for the occurrence of Braess' paradox in a general network, Steinberg and Zangwill assume that all links that carry flow before the network expansion also carry flow after the network expansion. They point out that if this assumption does not hold, however, this does not imply that the paradox will not occur, as it could conceivably occur

Table 2. Summary of analytical and numerical results

Situation	General expression	Braess' example (Braess, 1968)	Arnott and Small's example (Arnott and Small, 1994)
Braess' paradox does not occur because demand is too low.	$Q \leq \frac{2(\alpha_1 - \alpha_2)}{3\beta_1 + \beta_2}$	$Q \leq 2.58$	$Q \leq 500$
Braess' paradox occurs, but network expansion leads to improved system performance under marginal cost pricing.	$\frac{2(\alpha_1 - \alpha_2)}{3\beta_1 + \beta_2} < Q < \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}$	$2.58 < Q < 4.44$	$500 < Q < 750$
Braess' paradox occurs, and network expansion does not lead to improved system performance even under marginal cost pricing.	$\frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2} < Q < \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}$	$4.44 < Q < 8.89$	$750 < Q < 1500$
Braess' paradox does not occur because demand is too high.	$Q \geq \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}$	$Q \geq 8.89$	$Q \geq 1500$

under other assumptions. Our results, while not obtained for the case of a general network, show that Braess paradox can occur even if all links that carry flow before the network expansion do not carry flow after the expansion. Specifically, from eqn (5) we know that the original links carry no traffic when:

$$Q \leq \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2},$$

while eqn (12) tells us that Braess' paradox occurs when

$$Q \leq \frac{2(\alpha_1 - \alpha_2)}{3\beta_1 + \beta_2}. \tag{18}$$

Now, it is easy to show that there is a finite range of values for Q in which Braess' paradox occurs (eqn (18) is satisfied), but the original links carry no traffic (eqn (5) is also satisfied). Therefore, it is not necessary for the original links to carry traffic in order for Braess' paradox to occur. For example, for the numerical values in Arnott and Small's paper, Braess paradox occurs for $Q > 500$, but the additional link carries all the traffic for $Q \leq 750$. Thus, in this case, Braess' paradox occurs, although the original links carry no traffic, when $500 < Q \leq 750$.

4. CONCLUSIONS

The now well-known phenomenon in which addition of a link to a transportation network can leave all users worse off than before the link is added is commonly known as Braess' paradox, in recognition of the person who first described the phenomenon in a paper published in 1968. However, it is no longer appropriate to refer to this phenomenon as a paradox because we now know the cause of the unexpected result. Specifically, under average cost pricing travelers choose their routes according to their own best interest (leading to the user equilibrium flow pattern in the network), rather than taking into account the effects of their choices on others in the system (which would lead to the system optimal flows in the network). The latter flow pattern can, of course, be achieved by charging users the marginal cost of their travel, and it ensures that expansion of the network leaves users no worse off than before the expansion.

This paper employs the classical network configuration used to illustrate Braess' paradox to show that whether the paradox occurs or not is dependent on the parameters of this problem (the link congestion functions and the demand for travel). In particular, this paper shows that, for a given set of link congestion function parameters, Braess' paradox occurs only if the total demand for travel lies within a certain range of values (the bounds of which are dependent on the link congestion function parameters). That is, if the demand for travel is less than some lower bound or exceeds some upper bound, then the paradox does not occur. When the demand exceeds the upper bound, no one uses the new link because the advantage presented by the low free flow travel time is nullified by the increased travel time (at high volumes) on the bottleneck portions of the

network. The fact that there is some upper bound on the total demand, given the link congestion function parameters, means that a network might 'grow out' of the seemingly paradoxical situation as the demand increases over time. (This paper shows that for the example problems used by Braess and by Arnott and Small, demands somewhat higher than those in their examples would prevent the occurrence of the paradox.) More importantly, a network in which the paradox does not occur at a given level of demand might 'grow into' the paradox as demand increases over time.

It is well known that charging users the marginal cost of travel (and thus ensuring that the flow pattern is system optimal) guarantees that addition of a new link to an existing system will not increase the total system travel time; in a system optimal assignment, one can always ensure that the new link carries zero flow by charging a prohibitive toll on the new link. Of course, if it is best for the new link to carry no traffic in order to avoid a deterioration of the system performance, it would clearly make no sense to build the new link. On the other hand, if the new link carries some traffic and can improve system performance under marginal cost pricing, then a full examination of the costs and benefits of system expansion needs to be undertaken before a decision can be made as to whether to build the new link or not.

This paper shows that whether the new link carries any traffic in the system optimal flow pattern, and therefore whether the addition of the new link should be examined further, depends on the problem parameters. Specifically, the paper shows that, given the link congestion function parameters, if the demand exceeds some upper bound then the new link will be assigned no traffic in the system optimal flow pattern. In this situation, the new link does not improve the performance of the network (even under marginal cost pricing), and the network expansion is not warranted.

Finally, we note that all the results reported in this paper have been derived in the context of the classical, symmetric four-link network used by Braess to illustrate the phenomenon that now bears his name. Similarly, we have assumed linear link congestion functions and inelastic demand, in common with the classical formulation for illustrating Braess' paradox. The extension of these results to the general case awaits investigation.

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