Aggregative Public Good Games

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Abstract

We exploit the aggregative structure of the public good model to provide a simple analysis of the voluntary contribution game. In contrast to the best response function approach, ours avoids the proliferation of dimensions as the number of players is increased, and can readily analyze games involving many heterogeneous players. We demonstrate the approach at work on the standard pure public good model and show how it can analyze extensions of the basic model.

1. Introduction

The voluntary contribution model of pure public good provision has long been a favorite topic for students of public economics. Economists use it routinely to examine the inefficiency of decentralized resource allocation processes in the presence of externalities and to explore the properties of alternative mechanisms. Yet many existing analyses do not fully exploit its simplicity. The present paper develops an alternative way of analyzing the model that exploits its *aggregative* nature—that is, the fact that individuals care about, not an arbitrary vector of individual contributions, but a very

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specific aggregate, their unweighted sum. This feature has, of course, been noted by others—for example, Danziger (1976), Okuguchi (1993), Corchon (1994), and Cornes, Hartley, and Sandler (1999) have, to varying extents, sought to exploit it. The present paper develops the approach exploited by Cornes, Hartley, and Sandler, both simplifying it and also extending it to analyze comparative static questions.

Our approach has several virtues. It yields a unified analysis of existence, uniqueness and comparative static properties of the voluntary contribution model without requiring the use of fixed point or other theorems in high—dimensional spaces. Second, it also suggests a transparent geometric representation. A third and, in our view, decisive virtue of the approach is its power as a tool for the analysis of issues that are widely perceived to be not only theoretically interesting but also empirically significant. For example, the recent literature on "Global Public Goods" stresses the importance of productivity differences between countries as generators of global public goods—see, in particular, Arce and Sandler (1999), Sandler (1997), and the papers in Kaul et al. (1999, 2003). Yet the best response function method of analysis struggles to cope with such an extension of the basic model in settings that involve more than two players. Consequently, existing analyses—of which Ihori (1996) is a notable example—typically confine themselves to two-player games. By contrast, productivity differences across many potential contributors create no problems for the present approach. Finally, the public good game is only one of many with aggregative structure—other applications include oligopoly, contests, open access resource exploitation, and cost and surplus sharing models.

2. The Pure Public Good Model

2.1. Assumptions of the Model

There are n players. Player i's preferences are represented by the utility function

$$u_i = u_i(y_i, Q), \tag{1}$$

where y_i is the quantity of a private good and Q is the total quantity of a pure public good. We impose the following assumptions.

A1. *Well-behaved preferences:* For all i, the function $u_i(\cdot)$ is everywhere strictly increasing and strictly quasiconcave in both arguments. It is also everywhere continuous.[1]

A2. *Linear individual budget constraints:* Player i's budget constraint requires that

[1]Note that we dispense with the usual assumption of differentiability of the utility function.

$$y_i + c_i q_i \leqq m_i, \tag{2}$$

where $q_i \geqq 0$ is her contribution to a pure public good. The unit cost c_i and income m_i are strictly positive and exogenous.

A3. *Additive social composition function:* The total supply of the public good is the sum of all individual contributions:

$$Q = \sum_{j=1}^{n} q_j = q_i + Q_{-i}, \tag{3}$$

where Q_{-i} is the sum of the contributions made by all players except i.

Remark 1: The possibility of differing unit costs across players represents a significant extension of the basic pure public good model as set out in, say, Cornes and Sandler (1985) and Bergstrom, Blume, and Varian (1986). Our analysis shows that this extension neither complicates analysis nor threatens existence of a unique equilibrium. However, it has interesting comparative static implications.

The budget constraint (2) may be written so as to incorporate the contributions of others explicitly as a component of player i's income endowment. Add the quantity Q_{-i} to both sides. This yields

$$y_i + c_i Q \leqq m_i + c_i Q_{-i}. \tag{4}$$

This requires that the value of the bundle consumed by i cannot exceed that of her endowment point. This is her "full income," $M_i \equiv m_i + c_i Q_{-i}$. Furthermore, she is restricted to allocations consistent with the condition that $y_i \leqq m_i$, reflecting the assumption that she cannot undo the contributions of others and transform them into units of the private consumption good.

Player i chooses nonnegative values of y_i and q_i to maximize utility subject to her budget constraint and the prevailing value of Q_{-i}. To any nonnegative value of Q_{-i}, there corresponds a unique utility-maximizing contribution level, $\hat{q}_i$. By varying Q_{-i} parametrically, we generate her best response function, $\hat{q}_i = b_i(Q_{-i})$. At a Nash equilibrium, every player's choice is a best response to the prevailing choices of all other players.

Figure 1 depicts an individual's preferences and constraint set. The values of Q_{-i} and m_i fix the endowment point E, and the constraint set is the area ODEF, where the slope of EF is $-c_i$, reflecting the marginal rate of transformation between q_i and y_i. Strict quasiconcavity of $u_i(\cdot)$ implies a unique utility-maximizing response, shown as the point of support T. The figure also shows the locus of such points traced out for a given value of m_i as Q_{-i} varies parametrically. The figure shows this locus to be everywhere upward-sloping. This reflects the following assumption that we impose on preferences:

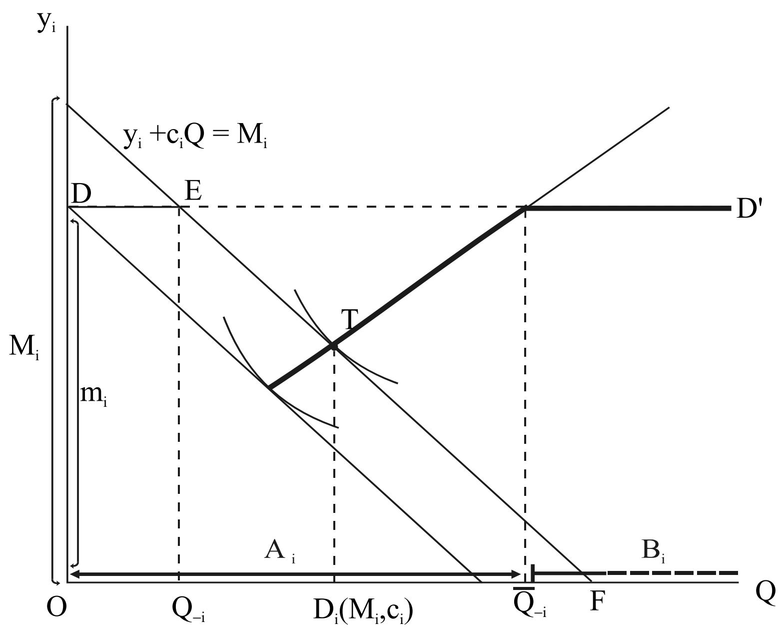

Figure 1: Player *i*'s preferences and constraint set

A4. *Normality:* For every player i, both the private good and the public good are normal. That is, the locus of values of y_i and Q consistent with a given marginal rate of substitution has positive finite slope everywhere.

2.2. The Replacement Function

2.2.1. The Individual Replacement Function

Define player i's demand function for the public good as $D_i(M_i, c_i)$. Since c_i is being held constant, we can focus on the role of income and write her demand as $\xi_i(M_i)$. This defines her Engel curve. Normality implies that ξ_i is increasing in M_i. Thus its inverse, $\xi_i^{-1}(Q)$, is well defined on the range of ξ_i.

We must respect the fact that the player cannot undo the contributions of others and enjoy a public good quantity that falls short of the total contributions of others. Define the following sets:

$$\mathbf{A}_i = \{Q_{-i} \,|\, \xi_i(m_i + c_i Q_{-i}) - Q_{-i} \geq 0\}$$
$$\mathbf{B}_i = \{Q_{-i} \,|\, \xi_i(m_i + c_i Q_{-i}) - Q_{-i} < 0\}.$$

Normality implies that $\xi_i(m_i + c_i Q_{-i}) - Q_{-i}$ is a decreasing function of Q_{-i} for all $Q_{-i} > 0$. Therefore the sets $\mathbf{A}_i$ and $\mathbf{B}_i$ are intervals on the real line of the form $[0, \bar{Q}_i]$ and $(\bar{Q}_i, \infty)$, respectively. Our assumptions do not rule out the possibility that $\bar{Q}_i = \infty$.

We now define the function $r_i(Q, m_i, c_i)$ as follows. If $Q_{-i} \in \mathbf{B}_i$, then $r_i(Q, m_i, c_i) = 0$. If $Q_{-i} \in \mathbf{A}_i$, then $r_i(Q, m_i, c_i) = \hat{r}_i$, where $\hat{r}_i$ is the unique solution to the equation $Q = \xi_i[m_i + c_i(Q - \hat{r}_i)]$. Taking the inverse of ξ_i, and rearranging terms, we find that if $Q_{-i} \in \mathbf{A}_i$,

$$r_i(Q, m_i, c_i) = \frac{m_i - \xi_i^{-1}(Q)}{c_i} + Q. \tag{5}$$

We need to check that the function $r_i(Q, m_i, c_i)$ is well defined and continuous in Q, and we will show that it is monotone decreasing if $Q_{-i} \in \mathbf{A}_i$. Define the function $\Phi_i(r_i) \equiv \xi_i[m_i + c_i(Q - r_i)] - Q$. For $Q \in \mathbf{A}_i$, $\Phi_i(0) = \xi_i[m_i + c_i(Q)] - Q \geq 0$ and $\Phi_i(Q + m_i/c_i) = \xi_i[0] - Q \leq 0$. Since ξ_i is a continuous increasing function, Φ_i is a continuous decreasing function. Hence there is precisely one solution for the equation, $\Phi_i(\hat{r}_i) = 0$. This justifies our claim above that $\hat{r}_i$ is the unique solution to the equation $Q = \xi_i[m_i + c_i(Q - \hat{r}_i)]$. Continuity of r_i when Q_{-i} is in the interior of $\mathbf{A}_i$ follows from the assumed continuity of the Engel curve. Furthermore, in the case where $\bar{Q}_i$ is finite, it follows from the continuity of the function $\xi_i[m_i + c_i Q] - Q$ and the definition of $\mathbf{A}_i$ that $\xi_i[m_i + c_i \bar{Q}_i] - \bar{Q}_i = 0$. Hence $r_i(Q, m_i, c_i)$ is continuous at $Q = \bar{Q}_i$. Finally, normality of the private good implies that Φ_i falls as Q increases, and hence that $r_i(Q, m_i, c_i)$ is decreasing in Q.

DEFINITION 1: *The function*

$$r_i(Q, m_i, c_i) \equiv \max\left\{\frac{m_i - \xi_i^{-1}(Q)}{c_i} + Q, 0\right\}$$

is player i's replacement function.

Remark 2: We call $r_i(Q, m_i, c_i)$ player i's replacement function for the following reason. Consider any Q in the domain for which $r_i(\cdot)$ is defined. Then there is a unique quantity $Z \in [0, Q]$ such that, if the amount Z were subtracted from the quantity Q, the player's best response to the remaining quantity would exactly replace the quantity removed, and $Z = r_i(Q)$.

Denote by $\underline{Q}_i$ the quantity that is player i's best response when all other players' contributions are zero. At such an allocation, player i's contribution is the total provision level: $r_i(\underline{Q}_i, m_i, c_i) = \underline{Q}_i$. Note that $Q_{-i} \in A_i$ is equivalent to $Q \in [\underline{Q}_i, \bar{Q}_i]$.

The following proposition summarizes the significant properties of the function $r_i(Q, m_i, c_i)$.

PROPOSITION 2.1: *If assumptions A1, A2, A3, and A4 hold, player i has a replacement function* $r_i(Q, m_i, c_i)$ *with the following properties:*

1. *There exists a finite value,* $\underset{\sim}{Q}_i$, *at which* $r_i(\underset{\sim}{Q}_i, m_i, c_i) = \underset{\sim}{Q}_i$.
2. $r_i(Q, m_i, c_i)$ *is defined for all* $Q \geqq \underset{\sim}{Q}_i$.
3. $r_i(Q, m_i, c_i)$ *is continuous.*
4. $r_i(Q, m_i, c_i)$ *is everywhere nonincreasing in Q, and is strictly decreasing wherever it is strictly positive.*

Remark 3: We call the value $\underset{\sim}{Q}_i$ player *i*'s *standalone value*. It is the level of the public good that player *i* would contribute if she were the sole contributor.

2.2.2. The Aggregate Replacement Function

We now define the aggregate replacement function of the game, $R(Q)$.

DEFINITION 2: *The aggregate replacement function of the game* $R(Q, \mathbf{m}, \mathbf{c})$ *is defined as*

$$R(Q, \mathbf{m}, \mathbf{c}) = \sum_{j=1}^{n} r_j(Q, m_j, c_j),$$

where $\mathbf{m} \equiv (m_1, m_2, \ldots, m_n)$ *and* $\mathbf{c} \equiv (c_1, c_2, \ldots, c_n)$.

The properties of the individual replacement functions are either preserved or else are modified in very slight and straightforward ways by the operation of addition. The following proposition summarizes the salient properties of $R(\cdot)$. All play a role in subsequent analysis.

PROPOSITION 2.2: *If assumptions A1–A4 hold for all i, there is an aggregate replacement function,* $R(\cdot) \equiv \sum_j r_j(\cdot)$, *with the following properties*:

1. $R(\max\{\underset{\sim}{Q}_1, \underset{\sim}{Q}_2, \ldots, \underset{\sim}{Q}_n\}, \mathbf{m}, \mathbf{c}) \geqq \max\{\underset{\sim}{Q}_1, \underset{\sim}{Q}_2, \ldots, \underset{\sim}{Q}_n\}$.
2. $R(Q, \mathbf{m}, \mathbf{c})$ *is defined for all* $Q \geqq \max\{\underset{\sim}{Q}_1, \underset{\sim}{Q}_2, \ldots, \underset{\sim}{Q}_n\}$.
3. $R(Q, \mathbf{m}, \mathbf{c})$ *is continuous.*
4. $R(Q, \mathbf{m}, \mathbf{c})$ *is everywhere nonincreasing in Q, and is strictly decreasing wherever it is strictly positive.*

Use of $R(\cdot)$ suggests a simple way of describing Nash equilibrium. A Nash equilibrium is an allocation at which every player is choosing her best response to the choices made by all other players. Clearly, the Nash equilibrium level of total provision, Q^N, must equal the sum of all best responses associated with the equilibrium allocation:

$$\hat{q}_1 + \hat{q}_2 + \cdots + \hat{q}_n = Q^N.$$

We have just shown that each best response may be described by that player's replacement function. A Nash equilibrium, therefore, may be characterized in the following way.

Characterization of Nash equilibrium: A Nash equilibrium is a strategy profile $\hat{\mathbf{q}}$ such that

$$\hat{q}_j = r_j(Q^N, m_j, c_j), \text{ for } j = 1, 2, \ldots, n, \text{ where } Q^N = \sum_{j=1}^{n} \hat{q}_j.$$

Remark 4: Note that Q^N is an equilibrium level of total provision of the public good if and only if $R(Q^N, \mathbf{m}, \mathbf{c}) = Q^N$. Once Q^N is known, individual choices can be read off using the replacement functions.

Remark 5: This characterization does not require a proliferation of dimensions as the number of players increases. One simply adds functions, each defined on an interval of the real line.

3. Nash Equilibrium: Existence and Uniqueness

Recall that a Nash equilibrium is an allocation at which $R(Q, \mathbf{m}, \mathbf{c}) = Q$. Referring back to Proposition 2.2, Property 1 locates a value in the domain of $R(\cdot)$ for which $R(\cdot) \geqq Q$. Property 2 identifies the domain on which $R(\cdot)$ is defined. Properties 3 and 4 guarantee the existence of a unique value, Q^*, at which $R(Q^*, \mathbf{m}, \mathbf{c}) = Q^*$. We can immediately infer the following.

PROPOSITION 3.1: *There exists a unique Nash equilibrium in the pure public good game.*

Example 1: Suppose that player i has the Cobb–Douglas utility function $u_i = \alpha_i \ln y_i + (1 - \alpha_i) \ln Q$ and money income m_i. Then, working through the standard first-order conditions for utility maximization, we get $\xi_i(M) = \frac{1-\alpha_i}{c_i} M_i$ and, therefore, $\xi_i^{-1}(Q) = \frac{c_i Q}{1-\alpha_i}$. Thus i's replacement function is

$$r_i(Q) = \text{Max}\left\{\frac{m_i - \xi_i^{-1}(Q)}{c_i} + Q, 0\right\} = \text{Max}\left\{\frac{m_i}{c_i} - \frac{\alpha_i Q}{1-\alpha_i}, 0\right\}.$$

In an economy of such individuals, the Nash equilibrium is found by solving

$$\sum_{i \in C} \frac{m_i}{c_i} - \frac{\alpha_i Q}{1-\alpha_i} = Q,$$

where C is the set of positive contributors at the equilibrium.

Figure 2 depicts the equilibrium in such an economy consisting of four individuals, under the assumptions that, for all $i, \alpha_i = \frac{1}{2}$ and $c_i = 1$. The

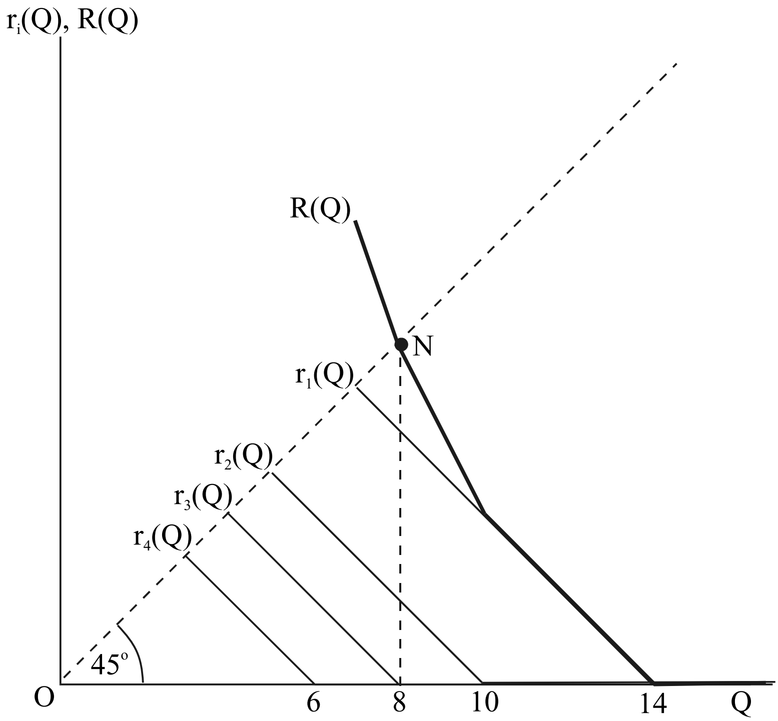

Figure 2: Individual and aggregate replacement functions

initial incomes are $(m_1, m_2, m_3, m_4) = (14, 10, 8, 6)$. The Nash equilibrium is the unique point of intersection between the graph of $R(Q)$ and the ray through the origin O with slope 1. At the equilibrium, two of the four players are positive contributors: $Q^N = 8$ and $(r_1, r_2, r_3, r_4) = (6, 2, 0, 0)$. Note that existence and uniqueness are effectively established by a single line of argument that exploits the continuity and monotonicity of the graph of $R(Q)$. Furthermore, differences across players with respect to preferences, income levels, and unit costs in no way complicate the analysis or its exposition.

4. Nash Equilibrium: Comparative Static Properties

We now analyze equilibrium responses to exogenous changes in players' incomes or unit cost levels. We model the effect of such shocks on equilibrium by first considering how they shift the graphs of individual replacement functions, and therefore the graph of $R(Q)$. The equilibrium value of Q rises, remains unchanged, or falls according to whether, at its initial equilibrium value, the aggregate replacement value rises, remains unchanged, or falls. The present approach allows us to consider finite shocks.

4.1. Comparative Statics of a Player's Replacement Function

4.1.1. Corner Solutions

Figure 2 depicts an example in which every player's replacement function falls to zero at some finite value of Q. Our assumptions up to this point do not necessarily imply that replacement functions have this property. Denote by $\psi_i(Q, c_i)$ the function that uniquely determines individual i's most preferred level of private good consumption as a function of Q and c_i. Assumptions A1–A4 are consistent with the possibility that, as Q increases, the value of $\psi_i(\cdot)$ asymptotically approaches some value $\bar{y}_i < m_i$. Her replacement function then does not fall to zero, but converges to some positive value as Q increases.

We have not committed ourselves on whether or not there is a finite value of Q at which $r_i(Q) = 0$ simply because our analysis is not at all complicated, and our conclusions concerning existence and uniqueness are not affected, by our answer to this question. However, the existing literature typically assumes—sometimes explicitly, sometimes not—the existence of such a value. In what follows, we adopt the following slightly stronger normality assumption.[2]

A4*. *Bounded normality:* For every player i, there is a finite value of Q, $\bar{Q}_i$, such that $\psi_i(\bar{Q}_i, c_i) = m_i$.

We will call the quantity $\bar{Q}_i$ player i's *dropout value*, since it is the value of total provision above which she drops out of the set of positive contributors: $r_i(Q, m_i, c_i) = 0$ for all $Q \geqq \bar{Q}_i$. Let the equilibrium level of aggregate provision be Q^N. Then any player whose dropout value falls short of Q^N will be a noncontributor at that equilibrium.[3]

4.2. Comparative Statics of Equilibrium Provision

Our comparative static properties are all consequences of the following three relationships, each of which has been shown above to hold at any allocation at which player i is choosing her best response to the contributions of others:

$$\hat{y}_i = \min\{\psi_i(Q, c_i), m_i\}, \tag{6}$$

$$\hat{q}_i = r_i(Q, m_i, c_i) = \frac{m_i - \hat{y}_i}{c_i} = \max\left\{\frac{m_i - \psi_i(Q, c_i)}{c_i}, 0\right\}, \tag{7}$$

and

$$u_i(\hat{y}_i, Q) = u_i(\min\{\psi_i(Q, c_i), m_i\}, Q). \tag{8}$$

[2]Andreoni (1988) makes an assumption that has essentially the same implication as A4*.
[3]Andreoni and McGuire (1993) note the significance of players' dropout values in determining who are in the set of positive contributors at a given equilibrium. More recently, McGuire and coauthors have exploited this magnitude in further exploration of the characteristics of equilibrium—see McGuire and Groth (1985) and McGuire and Shrestha (2003).

Before considering the effects of parametric changes in incomes and unit costs on equilibrium, we note two direct implications of these relationships.

PROPOSITION 4.1: *Let players i and j be positive contributors in equilibrium. Suppose further that they have identical preferences and identical unit cost coefficients. Then their equilibrium consumption bundles and utility are identical, even if their incomes differ.*

If player i is a positive contributor, then $\psi_i(Q, c_i) < m_i$. Then equation (6) implies that $\hat{y}_i = \psi_i(Q, c_i)$ and equation (8) implies equality of utilities. In short, a higher income contributor is no better off than an otherwise identical lower income contributor. Of course, income helps to determine whether a player contributes. But the fates of contributors are tied together independent of their incomes.

PROPOSITION 4.2: *Let players i and j be positive contributors in equilibrium. Suppose further that they have identical preferences, but differ with respect to their unit costs. Then at equilibrium,*

$$c_i > c_j \Longrightarrow u(\hat{y}_i, Q) > u(\hat{y}_j, Q).$$

Proof: Let $c_i > c_j$. At any given level of public good provision Q^*, $\psi(Q^*, c_i) > \psi(Q^*, c_j)$. Therefore, if both contribute, player i enjoys the higher level of private good consumption. Both enjoy the same level of public good provision, Q^N. Therefore, at the Nash equilibrium,

$$u\big(\psi\big(Q^N, c_i\big), Q^N\big) > u\big(\psi\big(Q^N, c_j\big), Q^N\big). \quad \blacksquare$$

In short, higher cost contributors are better off than otherwise identical lower cost contributors.[4] It does not pay to have a comparative advantage as a producer of the public good. As with Proposition 4.1, incomes matter only insofar as they determine whether or not a player is a positive contributor.

4.3. Comparative Statics of Income Changes

4.3.1. Income Changes with Idiosyncratic Unit Costs

Let C denote the set of positive contributors at a Nash equilibrium. Consider the equilibrium response by contributor i to a change in her income. The following proposition follows immediately from Equation (7).

[4]In a model in which the contributors to a public good are interpreted as countries, Boadway and Hayashi (1999) argue that more populous countries are worse off than less populous countries. Their mechanism works through the implications of population size for per capita cost of the public good.

PROPOSITION 4.3: *Let player i be a positive contributor both before and after an exogenous change in money income from m_i^0 to m_i^1. Then, at unchanged $\tilde{Q}$, the change in her replacement value is*

$$r_i(\tilde{Q}, m_i^1) - r_i(\tilde{Q}, m_i^0) = \frac{m_i^1 - m_i^0}{c_i}.$$

Starting at a Nash equilibrium, consider the effect on equilibrium of income redistribution among contributors. Assume that the set of positive contributors is not changed by the redistribution. At the initial equilibrium provision level, Q^*, the value of the aggregate replacement function rises, stays unchanged, or falls according to whether

$$\sum_{j \in C} \frac{(m_j^1 - m_j^0)}{c_j} >, =, \text{ or } < 0.$$

For a given set of incomes, the aggregate replacement function is nonincreasing in Q. Therefore the following corollary of Proposition 4.3 holds.

COROLLARY 4.1: *Consider a set of changes in the incomes of contributors. If the set of positive contributors is unchanged, then aggregate equilibrium provision rises, remains unchanged or falls according to whether* $\sum_{j \in C} \frac{(m_j^1 - m_j^0)}{c_j} >, =,$ *or* <0.

For example, a transfer from Contributor A to Contributor B increases equilibrium provision if $c_A > c_B$. Redistribution from a higher to a lower cost contributor enhances efficiency, and the efficiency gain is partly taken through an increase in the provision of the public good.

Not only does such a redistribution increase equilibrium public good provision—it is also Pareto improving.[5] The reasoning is simple. Each individual's preference map in (y, Q) space is fixed throughout the present thought experiment. Under the normality assumption, if each individual is enjoying a higher level of total public good provision after the redistribution, she must have moved upwards and to the right along her income expansion path. Hence, her consumption of the private good is higher, and so must be her utility. In short,

COROLLARY 4.2: *If changes in contributors' incomes leave the set of positive contributors unchanged, the new equilibrium is Pareto superior to, identical with, or Pareto inferior to the initial equilibrium according to whether* $\sum_{j \in C} \frac{(m_j^1 - m_j^0)}{c_j} >, =,$ *or* <0.

[5]Within the context of a two-player model, Buchholz and Konrad (1995) note that transfers between contributors with different productivities may benefit the donor. They also explore the incentive this provides for individuals to make transfers prior to playing the contribution game. Buchholz, Konrad, and Lommerud (1997) further explore two-stage contribution games involving two players. Both analyses exploit the aggregative nature of the public good model.

The following implication of differences in the unit cost of public good provision across contributors is worth noting. A redistribution may lead to a Pareto improving increase in public good provision even though it reduces the aggregate income of contributors. This follows from the simple observation that the inequalities $\sum_{j\in C}(m_j^1 - m_j^0) < 0$ (a reduction in the aggregate income of contributors) and $\sum_{j\in C}\frac{(m_j^1 - m_j^0)}{c_j} > 0$ (a Pareto improving change in contributors' incomes) are perfectly consistent with one another if unit costs vary across individuals. Conversely, of course, an increase in the aggregate income of contributors is consistent with a reduction in the equilibrium level of provision.

4.3.2. Income Changes with Common Unit Costs

Consider the standard model of public good provision with common unit costs across contributors—let $c_i = c_j \; \forall i, j \in C$. Then the well-known neutrality property follows as a direct corollary of Proposition 4.3.

COROLLARY 4.3: *Assume that $c_i = c_j \; \forall i, j \in C$. Then a pure redistribution of income among contributors—that is, a set of transfers such that $\sum_{j\in C} m_j^1 = \sum_{j\in C} m_j^0$—that leaves the set of positive contributors unchanged has no effect on the equilibrium allocation.*

The neutrality property implies that, in a well-defined sense, the set of positive contributors who face the same unit cost of public good provision behaves like a single individual. If attention is confined to income distributions that are consistent with a given set of positive contributors, then the aggregate replacement function associated with that set depends upon just two arguments: the total income of all contributors, and the value of Q.

COROLLARY 4.4: *For all income distributions consistent with a given set C of players being the positive contributors to the public good in equilibrium, $R(\cdot) = R(Q, M_C)$, where $M_C = \sum_{j\in C} m_j$.*

Consider the response of total equilibrium public good provision to a change in the total income received by the set of contributors. We assume throughout that the set of contributors is unchanged. At a Nash equilibrium,

$$R(Q^N) = \sum_{j\in C} r_j\big(Q^N, m_j\big) = Q^N. \tag{9}$$

Now suppose that the contributors' income levels change. At the new equilibrium, it remains the case that the sum of replacement values equals the total provision. Differentiating Equation (9),

$$\sum_{j\in C} r_{jQ}(\cdot)\, dQ^N + \sum_{j\in C} r_{jm}(\cdot)\, dm_j = dQ^N. \tag{10}$$

We have already shown that $r_{jm}(\cdot) = 1$, for all $j \in C$. Writing $M_C \equiv \sum_{j \in C} m_j$, Equation (10) becomes

$$\sum_{j \in C} r_{jQ}(\cdot)\, dQ^N + dM_C = dQ^N,$$

or

$$\frac{dQ^N}{dM_C} = \frac{1}{1 - \sum_{j \in C} r_{jQ}(\cdot)}. \tag{11}$$

This is precisely the result obtained by Cornes and Sandler (2000). To get a better feeling for the magnitude of this response, let contributors be identical, with $r_{iQ} = r_{jQ} = r_Q$, say, for all $i, j \in C$. Then Equation (11) becomes

$$\frac{dQ^N}{dM_C} = \frac{1}{1 - n_C r_Q(\cdot)}. \tag{12}$$

Normality implies that $-\infty < r_Q(\cdot) < 0$. If we suppose that r_Q is bounded away from zero—that is, there is some value ϵ such that $r_Q(\cdot) \leqq \epsilon < 0$—then (12) implies that

$$\lim_{n_C \to \infty} \frac{dQ^N}{dM_C} = 0.$$

For example, suppose that each has a Cobb–Douglas utility function of the form $u_i(y_i, Q) = yQ^\alpha$. Then $r_i(\cdot) = m_i - Q/\alpha$, $r_Q(\cdot) = -1/\alpha$ and $\frac{dQ^N}{dM_C} = \frac{\alpha}{\alpha + n_C}$. Suppose $\alpha = 1$. Then, if $n = 10$, $dQ^N/dM_C = 1/11$. If $n_C = 100$, $dQ^N/dM = 1/101$. For a given common value of the individual marginal propensity, the magnitude of the aggregate propensity falls rapidly as n_C increases.

A further implication of Equation (11) is worth noting. Suppose that the existing contributors are not identical. Then

$$\frac{dQ^N}{dM} = \frac{1}{1 - \sum_{j \in C} r_{jQ}(\cdot)} \leqq \frac{1}{1 - n_C \min_{j \in C}\{r_{jQ}\}} < \frac{1}{1 - \min_{j \in C}\{r_{jQ}\}}. \tag{13}$$

This may be related to the slope of the relevant player's income expansion path. Contributor i's most preferred level of public good may be written as a function of her full income: $\hat{Q}_i(m_i + Q_{-i})$. We have the identity $\hat{Q}_i(m_i + Q_{-i}) = Q_{-i} + r_i(\hat{Q}_i(m_i + Q_{-i}), m_i)$. Differentiating and rearranging,

$$\frac{1}{1 - r_{iQ}(\cdot)} = \hat{Q}_i'.$$

That is, the aggregate response dQ^N/dM cannot exceed the smallest individual response, $\min_{j \in C}\{\hat{Q}_j'\}$. Not only does the interaction between players' responses dampen the response of aggregate provision to any change in the income of the set of positive contributors. In addition, the presence of just one contributor with a low propensity to contribute is enough to place a precise upper bound on the aggregate propensity to contribute of a given set of positive contributors.

What are the normative implications of a redistribution of initial incomes in this model? We have shown that redistributions of initial income among positive contributors change nothing. Redistributions among noncontributors benefit the recipients and hurt the donors, leaving the utilities of all others unchanged. But what about redistributions from noncontributors to contributors? Cornes and Sandler (2000) show that, even when every individual faces the same unit cost of contribution to the public good, such transfers can lead to a new Nash equilibrium that Pareto-dominates the equilibrium associated with the initial income distribution. This is easily shown in a simple two-type economy. Consider an equilibrium of a public goods economy at which there are n_N noncontributors and n_C positive contributors. The utility of a typical noncontributor is

$$u_N = u_N(y_N, Q) = u_N(m_N, Q).$$

Now suppose that the same amount of income is taken from each noncontributor and given to a positive contributor. To keep the exposition simple, assume that the set of contributors is unchanged at the new equilibrium. Let the total extra income received by all contributors be dM_C. Each noncontributor loses an amount of income $dm_N = -dM_C/n_N$.

The change in utility of a typical noncontributor is

$$\begin{aligned} du_N &= \frac{\partial u_N(m_N, Q)}{\partial y_N} dm_N + \frac{\partial u_N(m_N, Q)}{\partial Q} dQ. \\ &= \frac{\partial u_N(m_N, Q)}{\partial y_N} [\nu_N dQ + dm_N] \\ &= \frac{\partial u_N(m_N, Q)}{\partial y_N} [\nu_N dQ - dM_C/n_N], \end{aligned}$$

where $\nu_N \equiv \frac{\partial u_N(m_N, Q)/\partial Q}{\partial u_N(m_N, Q)/\partial y_N}$ is the noncontributor's marginal valuation of the public good. The fact that an individual is choosing not to contribute implies that, at equilibrium, $\nu_N < c$. This is consistent with her placing a strictly positive valuation on the public good. The typical noncontributor will be better off if, in the course of adjustment to the new equilibrium, $\nu_N dQ - dM_C/n_N > 0$.

To determine whether noncontributors are made better off, we need to determine the endogenous response of total provision. We already know that

$$dQ^N = \left\{ \frac{1}{1 - n_C r_Q(\cdot)} \right\} dM_C. \tag{14}$$

Substituting into (14),

$$\frac{du_N}{dM_C} = \frac{\partial u_N(m_N, Q)}{\partial y_N}\left[\nu_N\left(\frac{1}{1 - n_C r_Q(\cdot)}\right) - 1/n_N\right].$$

The right-hand side is positive if the expression in square brackets is positive—that is, if

$$\nu_N n_N > 1 - n_C r_Q(\cdot).$$

Using the result obtained above that $\frac{1}{1-r_{iQ}(\cdot)} = \hat{Q}'_i$, this condition may alternatively be expressed as

$$\hat{Q}' > \frac{n_C}{n_N \nu_N + n_C - 1}.$$

This makes sense. If n_N is large, each noncontributor is only one of many who are giving up income, and the gain in the aggregate income of contributors may be significant by comparison. Furthermore, the greater is $\hat{Q}'$, the greater is the additional public good provision purchased by a given transfer of income.

4.4. Comparative Statics of Changes in c_i

Let player i be a positive contributor at equilibrium. Now consider equilibrium responses to a discrete fall in her unit cost as a public good provider from its initial value of c_i^0 to c_i^1.

$$\begin{aligned} c_i^1 < c_i^0 &\Longrightarrow \psi_i(Q, c_i^1) < \psi_i(Q, c_i^0) \\ &\Longrightarrow r_i(Q, m_i, c_i^1) > r_i(Q, m_i, c_i^0). \end{aligned}$$

Evaluated at the initial Nash equilibrium, Q^{N_0}, the aggregate replacement value must now therefore exceed the equilibrium value. Therefore the equilibrium value of public good provision must rise: $Q^{N_1} > Q^{N_0}$.

The rise in equilibrium value of public good provision has an unambiguously beneficial effect for all players other than player i:

PROPOSITION 4.4: *If player i is a positive contributor, a reduction in c_i raises the equilibrium utility levels of all players other than i.*

However, the effect on the utility of contributor i herself is ambiguous. On the one hand, the increase in equilibrium provision is beneficial. However, for any given level of provision, player i finds herself contributing a higher equilibrium share.

PROPOSITION 4.5: *A reduction in c_i may either raise or reduce i's equilibrium level of utility.*

The only player not guaranteed an increase in utility is the one who enjoyed the exogenous reduction in unit cost as a contributor![6]

4.4.1. An Equal Proportional Change in All c_j's

Write player *i*'s unit cost as λc_i. Then an equal proportional change in each player's unit cost is modeled as a change in λ. Player *i*'s replacement function may be written as

$$r_i(Q, m_i, \lambda c_i) = \frac{m_i - \hat{y}_i}{\lambda c_i} = \text{Max}\left\{\frac{m_i - \psi_i(Q, \lambda c_i)}{\lambda c_i}, 0\right\}.$$

A reduction in λ shifts every contributor's replacement function upward, thereby increasing the equilibrium level of provision. By itself, the increase in Q increases the utility of each. However, we cannot conclude that every player is made better off. Recall player *i*'s utility function:

$$u_i(\hat{y}_i, Q) = u_i(\min\{\psi_i(Q, \lambda c_i), m_i\}, Q).$$

Player *i* enjoys a higher equilibrium level of Q. However, for any given level of Q, she also experiences a lower level of private good consumption. This effect may dominate her utility response. Another way to understand this possibility is to observe that, in the move to the new equilibrium, the total contributions made by players other than player *i* may fall. This harmful change in *i*'s endowment may outweigh the benefit of her own relative price effect.

PROPOSITION 4.6: *An equal proportional reduction in all contributors' unit costs will raise the equilibrium level of total provision. However, it may or may not raise the equilibrium level of utility enjoyed by contributor i.*

Here is an example of a situation in which an equal proportional reduction in every contributor's unit cost may make a contributor worse off.

Example 2: There are two individuals, whose utility functions are $u_1 = y_1^{3/4} + Q^{3/4}$, and $u_2 = \text{Min}\{y_2, Q\}$. Initially, $a_1 = a_2 = c_1 = c_2 = m_1 = m_2 = 1$. It may be confirmed that, in equilibrium,

$$(y_1, y_2, Q, u_1, u_2) = (0.67, 0.67, 0.67, 1.47, 0.67).$$

Now suppose that $c_1 = c_2 = 0.9$, while all other parameter values remain unchanged. The new equilibrium values are

$$(y_1, y_2, Q, u_1, u_2) = (0.46, 0.76, 0.78, 1.44, 0.78).$$

[6]Buchholz, Nett, and Peters (1998) extend the public good model to incorporate many heterogeneous players with endogenous incomes in an analysis that exploits its aggregative structure.

Player 1, with the high elasticity of substitution between the two goods, has responded by contributing significantly more to the public good, and is worse off. Player 2, by contrast, is now contributing less, and is better off.

If players are identical in every respect, so that the game is symmetric, the equilibrium response of the representative contributor's utility to a change in λ can be signed. In this case, every player's replacement function shifts upward in response to a reduction in λ. Hence total provision rises. So too must each player's contribution at the new equilibrium. Therefore each player experiences both an increase in her endowment of contributions by others and also a beneficial relative price effect.

PROPOSITION 4.7: *If all contributors are identical, an equal reduction in every contributor's unit cost raises both the equilibrium total provision of public good and the equilibrium level of utility enjoyed by the representative contributor.*

This last proposition was obtained by Cornes and Sandler (1989).

5. Concluding Comments

Aggregative games can be analyzed by conditioning the choices of individual players on the sufficient statistic that appears as an argument in the payoff of each. In contrast to the best response function, which conditions player i's choice on the choices of all players excluding player i, this simple trick avoids the unwelcome proliferation of dimensions as the number of heterogeneous players increases. It thereby permits a simple analysis of asymmetric aggregative games even with many players and the possibility of boundary solutions. It also lends itself to an elementary and revealing geometric representation.

Our discussion of these functions and their application to aggregative games has hardly scratched the surface of a potentially significant range of applications. To begin with, there are natural extensions to other public good models. We have elsewhere explored the implications of more general production processes whereby individual contributions generate the total quantity Q—see Cornes and Hartley (2001). The impure public good model, in which an activity jointly generates a private and a public characteristic, is also aggregative. Here, the change in a player's contribution that is consistent with a given change in aggregate provision is a little more complicated than in the pure public good model, depending as it does on patterns of substitutability and complementarity among the goods as well as on the conventional income term—see Cornes and Sandler (1994, 1996). But assumptions that imply a continuous decreasing replacement function again are sufficient to ensure the existence of a unique Nash equilibrium. Finally, we have shown elsewhere that the monotonicity of the replacement function, while sufficient, is not necessary, for equilibrium in an aggregative game to be unique.

A weaker sufficient condition is that for all i the function $s_i(Q) \equiv r_i(Q)/Q$ be decreasing wherever $r_i(Q) > 0$. This allows the private good to be inferior over some range without endangering uniqueness.

Aggregative games lie at the heart of many other economic models, including many Cournot and Bertrand games, contest theory, cost and surplus sharing models, and models of the "anticommons." The use of replacement functions offers the prospect of further insights in these and in many other applications. We have elsewhere extended our approach to model problems—such as cost and surplus sharing, and Tullock contests—in which both "best response" and "replacement" functions fail to be monotonic (see Cornes and Hartley 2000, 2003). We believe that the present approach merits serious attention as the most natural way to analyze aggregative games.

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