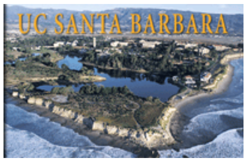

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Abstract:
This paper shows that if workers have identical wealths, abilities, and preferences then a draft lottery is Pareto superior to a voluntary army. It also shows that if being a civilian is a "normal good", then the optimal pay schedule will be such that people prefer not being chosen for the army. The paper shows how this idea extends to occupational choice in general and shows that pure gambles taken prior to occupational choice can substitute for lotteries that determine one's occupation. This paper repairs what I think is a major flaw in standard general equilibrium theory, which assumes away the nonconvexity of preferences that follows from the discreteness of occupational choice.

CHAPTER 2

Soldiers of fortune?

Theodore Bergstrom

1 Recruiting an army for a homogeneous country

A *The fortunes of soldiers in the absence of private lotteries or insurance*

Imagine a country where all N citizens have the same tastes and abilities. There is just one consumption good, bread. To defend itself, this country must raise an army of A soldiers. The other $N-A$ citizens are farmers who produce a total of W units of bread. Farmers are taxed to pay the soldiers. Let w_A be the wage rate paid to soldiers and w_F be the amount of bread left to each farmer after taxes. The net wages, w_A and w_F, must satisfy the social feasibility constraint

$$Aw_A+(N-A)w_F=W. \tag{1.1}$$

Equivalently, if we define $\bar{\pi}=A/N$ and $\bar{w}=W/N$,

$$\bar{\pi}w_A+(1-\bar{\pi})w_F=\bar{w}. \tag{1.2}$$

We will call (w_A, w_F) a *feasible wage structure* if (1.2) is satisfied.

Being a soldier is unpleasant and dangerous. If soldiers and farmers were paid the same wage, everyone would want to be a farmer. The country could offer high enough wages to soldiers to attract a volunteer army or it could select its army by lottery. A *fair draft lottery* would give everyone the same probability $\bar{\pi}=A/N$ of being drafted. For an arbitrarily chosen

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feasible wage structure, citizens will not be indifferent about whether or not they are selected for the army. But since everyone faces the same probability of being drafted, they will all have the same ex ante expected utility.

Let us assume that each citizen is an expected utility maximizer with a utility function

$$U(\pi, c_A, c_F) = \pi u_A(c_A) + (1-\pi) u_F(c_F), \tag{1.3}$$

where π is the probability that he will be in the army, c_A and c_F are the amounts of bread that he would consume in the army and on the farm, and $u_A(c_A)$ and $u_F(c_F)$ are smooth, increasing, strictly concave functions. In the absence of private lottery arrangements and insurance markets, it would have to be that $c_A = w_A$ and $c_F = w_F$. In this case, if there is a fair draft lottery with wage structure (w_A, w_F), then the utility of a representative citizen will be

$$U(\bar{\pi}, w_A, w_F) = \bar{\pi} u_A(w_A) + (1-\bar{\pi}) u_F(w_F). \tag{1.4}$$

We define the *optimal wage structure* to be the feasible wage structure that maximizes the expected utility of a representative citizen if the army is chosen by a fair lottery. The optimal wage structure (w_A^*, w_F^*) therefore maximizes equation (1.4) subject to the constraint (1.2). Simple calculus informs us that there is a unique solution for (w_A^*, w_F^*) and that, at this solution,

$$u_A'(w_A^*) = u_F'(w_F^*). \tag{1.5}$$

For any wage w paid to farmers, let us define $c(w)$ to be the *compensating wage differential* that would have to be paid to induce people to be soldiers when farmers are paid w. Since preferences are continuous and monotonic, $c(w)$ is a well-defined function. In general, $c(w)$ could be either positive or negative, depending on which occupation is more attractive, or even positive for some values of w and negative for others. It is also logically possible that for some wage rates of farmers no premium would be large enough to induce a person to voluntarily choose the army, in which case we define $c(w) = \infty$. Where it is finite valued, the compensating differential function $c(w)$ is defined implicitly by the equation

$$u_A(w + c(w)) = u_F(w). \tag{1.6}$$

If there is to be a volunteer army, then the prospect of being a soldier must be just as attractive as that of being a farmer. Let us assume that if soldiers got all the bread and farmers got none, then everyone would want to be a soldier, and that if farmers got all the bread and soldiers got none, then everyone would want to be a farmer. It then follows from continuity and monotonicity of preferences that there will be a unique feasible wage structure $(\hat{w}_A, \hat{w}_F)$ that is consistent with a volunteer army. The wage structure $(\hat{w}_A, \hat{w}_F)$ satisfies equation (1.2) and

$$\hat{w}_A = \hat{w}_F + c(\hat{w}_F) \tag{1.7}$$

so that

$$u_A(\hat{w}_A) = u_F(\hat{w}_F). \tag{1.8}$$

The utility level achieved by all citizens if there is a volunteer army would also be attained with a fair draft lottery in which the wage structure is $(\hat{w}_A, \hat{w}_F)$. This allows us to make the following simple, but useful, observation.

Proposition 1.1. If all citizens have identical preferences and if there are no private lotteries or insurance markets, then the expected utility of the representative citizen will be at least as high with a fair lottery, given the optimal wage structure, as it would be with a volunteer army.

In fact, except for a very special class of preferences, it will be the case that a fair lottery with an optimal wage structure dominates the volunteer army. The volunteer army wage structure equalizes *total* utilities $u_A(\hat{w}_A)$ and $u_F(\hat{w}_F)$, whereas, as we have shown, the optimal equalitarian wage structure equalizes *marginal* utilities $u'_A(w^*_A)$ and $u'_F(w^*_F)$. Only in the special case that the former condition implies the latter will the volunteer army be as good as a fair lottery with the optimal wage structure.

Two easy examples

Example 1.1. Consider a country where all citizens have identical von Neumann-Morgenstern utility functions such that for all w, $u_A(w) = u_F(w) - \alpha$, where $\alpha > 0$, $u'_F(w) > 0$, and $u''_F(w) < 0$. The optimal wage structure must satisfy $u'(w^*_A) = u'(w^*_F)$, which in this instance implies that $w^*_A = w^*_F$. Then $u_A(w^*) = u_F(w^*) - \alpha < u_F(w^*)$. Therefore, the optimal wage structure pays soldiers and farmers the same wage rate and leaves those who are chosen for the army worse off than those who are not.

Example 1.2. Suppose that all citizens have von Neumann-Morgenstern utility functions such that for all w, $u_A(w) = u_F(w - \alpha)$, where, as before

$\alpha > 0$, $u_F'(w) > 0$, and $u_F''(w) < 0$. Then $u_A'(w_A^*) = u_F'(w_F^*)$ implies that $w_A^* - \alpha = w_F^*$ and therefore $u_A(w_A^*) = u_F(w_F^*)$. In this case, the optimal wage structure makes farmers and soldiers equally well off. Therefore, the optimal wage structure is the same as the wage structure for the volunteer army.

There is a very simple condition that determines whether in a fair draft lottery with the optimal wage structure farmers will be better off than soldiers or vice versa. If the compensating differential function $c(w)$ is increasing in w, then the wage premium for the amenity of being a farmer rather than a soldier is an increasing function of wealth. It therefore is natural to make the following definition.

Definition. Farming is a *normal good at* w if $c'(w) > 0$ and an *inferior good at* w if $c'(w) < 0$.

Proposition 1.2. Let (w_A^*, w_F^*) be the optimal wage structure. With this wage structure, everyone will prefer being a farmer to being a soldier if and only if farming is a normal good at w_F^*. With the optimal wage structure, everyone will prefer being a soldier to being a farmer if and only if farming is an inferior good at w_F^*.

Proof: Differentiating both sides of the identity (1.6), one has

$$u_A'(w + c(w))(1 + c'(w)) = u_F'(w). \tag{1.9}$$

Therefore $u_A'(w_F^* + c(w_F^*)) < u_F'(w_F^*)$ if and only if $c'(w_F^*) > 0$. Since $u_A'(w_A^*) = u_F'(w_F^*)$ and since by assumption $u_A'(\cdot)$ is monotone decreasing, it must be that $w_A^* < w_F^* + c(w_F^*)$ if and only if $c'(w_F^*) > 0$. But

$$u_A(w_F^* + c(w_F^*)) = u_F(w_F^*)$$

and $u_A(\cdot)$ is monotone increasing in w. Therefore, $u_A(w_A^*) < u_F(w_F^*)$ if and only if $c'(w_F^*) > 0$. This proves the first assertion of Proposition 1.1. The second assertion is established by a similar argument. ■

Notice that, by our definition, the condition that determines whether farming is a normal good is a condition on the sign of the *derivative* of $c(w)$ and not on the sign of $c(w)$ itself. For example, if $c(w)$ were always negative and $c'(w)$ always positive, then it would be the case that at equal wages the army would be preferred to the farm, but with the optimal wage structure, farming would be preferred to the army.

B *A draft lottery with private draft insurance*

Suppose that the country selects its army by a fair draft lottery but chooses a feasible wage structure (w_A, w_F) different from (w_A^*, w_F^*). Then there is a reason for private markets in "draft insurance" to develop. A citizen who can buy actuarially fair insurance can afford to consume contingent consumptions c_A if drafted and c_F if not so long as

$$\bar{\pi}c_A + (1-\bar{\pi})c_F = \bar{\pi}w_A + (1-\bar{\pi})w_F = \bar{w}. \tag{1.10}$$

Since each citizen will choose (c_A, c_F) to maximize expected utility subject to equation (1.10), the problem solved is precisely the same as the problem we solved to find the optimal wage structure (w_A^*, w_F^*). Therefore we can assert the following.

Proposition 1.3. If the army is selected by a fair draft lottery with any feasible wage structure and if there are actuarially fair draft insurance markets, citizens will buy insurance so that their contingent consumptions are $c_A = w_A^*$ and $c_F = w_F^*$ where (w_A^*, w_F^*) is the wage structure of the best fair lottery.

From Proposition 1.3 we see that if the army is chosen by a fair lottery, then even if the government sets the "wrong" wage structure, private insurance contracts can "correct the mistake" so that after insurance contracts are settled, the contingent consumptions of farmers and soldiers are the same as the wages corresponding to the best fair draft lottery.

C *Is a draft lottery needed when fair financial lotteries are available?*

In an earlier draft of this chapter, it was argued that if all citizens had the same tastes and abilities, Propositions 1.1–1.3 would constitute a strong case for selecting an army by a draft lottery rather than having a volunteer army. Of course, in a world where preferences and abilities differ, there will be more reasons to use markets that sort people according to taste and comparative advantage.

Professor Oliver Hart suggested in private correspondence that this view is misleading because even with a volunteer army, it would be possible for private lotteries to achieve a Pareto-optimal allocation. It should be no surprise that efficiency could be achieved if there were private lotteries for ordinary goods as well as lotteries in which the "prizes" were obligations to join the army. But Professor Hart's claim is stronger. He argued that if the government offers the utility-equalizing wages $(\hat{w}_A, \hat{w}_F)$, then the availability of actuarially fair lotteries with prizes of ordinary goods is all that is necessary to enable individuals to achieve the utility level of a fair lottery with the optimum wage structure. [A recent article by Marshall (1984) advances a similar idea.] This turns out to be essentially correct and, at least to me, quite surprising.

Suppose that there is a volunteer army with the wage structure $(\hat{w}_A, \hat{w}_F)$ and let (w_A^*, w_F^*) be the optimal wage structure defined in the previous section. Consider a lottery in which with probability $\bar{\pi}$ the prize is $w_A^* - \hat{w}_A$ and with probability $1-\bar{\pi}$ the prize is $w_F^* - \hat{w}_F$. (From Proposition 1.2, it follows that if farming is a normal good, then the former prize will be positive and the latter negative.) Since the volunteer army wage structure and the optimal wage structure are both feasible, it must be that

$$\bar{\pi}(w_A^* - \hat{w}_A) + (1-\bar{\pi})(w_F^* - \hat{w}_F) = 0. \tag{1.11}$$

Therefore, this lottery is actuarially fair.

Now suppose the citizen participates in this lottery and adopts the following strategy: If the prize is $w_A^* - \hat{w}_A$, then he joins the army, and if the prize is $w_F^* - \hat{w}_F$ then he farms. With this strategy, the citizen will have a probability $\bar{\pi}$ of being in the army and consuming $\hat{w}_A + (w_A^* - \hat{w}_A) = w_A^*$ and a probability of $1-\bar{\pi}$ of being on the farm and consuming $\hat{w}_F + (w_F^* - \hat{w}_F) = w_F^*$. But this is precisely what the prospects would be if there were a fair draft lottery with the optimal wage structure. Furthermore, if all citizens choose this strategy, then the fraction $\bar{\pi}$ of the population will receive prizes that induce them to choose the army. This observation can be phrased as follows:

Proposition 1.4. If consumers have access to actuarially fair lotteries and if the wage structure fully compensates for the utility difference between being a farmer and being a soldier, then it is possible for consumers to replicate the prospects they would have with a fair draft lottery and the optimal wage structure by the device of making purely financial bets and conditioning their occupational choice on the outcome of the financial bets.

A skeptical reader may wonder whether the strategy that we have proposed for replicating the optimal wage structure by private lotteries is *self-enforcing* in the sense that those who "lose" the lottery (i.e., win the negative prize $w_A^* - \hat{w}_A$) will want to choose the army rather than the farm in the event that they lose the lottery. Similarly one may ask whether "winners" will choose the farm after they have learned the outcome of the lottery. To show that those who lose in the lottery will choose the army, we need to show that

$$u_A(\hat{w}_A+(w_A^*-\hat{w}_A)) \geq u_F(\hat{w}_F+(w_A^*-\hat{w}_A)). \tag{1.12}$$

From convexity of the function $u_F(\cdot)$ and equation (1.11), it follows that

$$\bar{\pi}u_F(\hat{w}_F+(w_A^*-\hat{w}_A))+(1-\bar{\pi})u_F(\hat{w}_F+(\hat{w}_F-\hat{w}_F)) \leq u_F(\hat{w}_F). \tag{1.13}$$

But, according to Proposition 1.1,

$$\bar{\pi}u_A(w_A^*)+(1-\bar{\pi})u_F(w_F^*) \geq u_F(\hat{w}_F). \tag{1.14}$$

Subtracting inequality (1.13) from (1.14) and simplifying terms, we have inequality (1.12). An exactly parallel proof establishes that

$$u_F(\hat{w}_F+(w_F^*-\hat{w}_F)) \geq u_A(\hat{w}_A+(w_F^*-\hat{w}_F)) \tag{1.15}$$

so that winners will always choose to farm.

The fact that the proposed strategy is self-enforcing is important because if this were not the case, then in order to achieve the utility level associated with this strategy, citizens would have to *precommit* themselves by means of some contract that forced them to join the army if they lost in the lottery. This would be contrary to the spirit of our discussion, since we sought a solution in which the only bets that citizens made were bets with financial prizes. As it turns out, although losers in the lottery may complain about their "bad luck" and may wish they had not bet, there is no better option for them, given that they have lost, than to "make up some of their losses" by joining the army.

But this is not quite the end of the story. We have shown that given the wage structure $(\hat{w}_A, \hat{w}_F)$, there is a strategy whereby citizens *can* achieve the expected utility generated by an optimal wage structure. We have *not* shown that this is the *expected utility-maximizing* strategy for citizens who have access to actuarially fair financial lotteries. In fact, this is in general not the case. With the wage structure $(\hat{w}_A, \hat{w}_F)$, citizens who are able to make any actuarially fair financial bets and to condition their choice of occupation on the outcome of the financial bet will not choose strategies that send them to the army with probability $\bar{\pi}$. Therefore, if citizens have access to all such bets, the wage structure $(\hat{w}_A, \hat{w}_F)$ will not be an "equilibrium" because it will *not* attract an army of expected size $\bar{\pi}$. Generally, however, there will exist *some* wage structure such that citizens using optimal strategies will join the army with probability $\bar{\pi}$. We pursue this matter in the next section.

D Probabilistic recruitment equilibrium with private lotteries

Suppose that the wage structure is (w_A, w_F) and that citizens can participate in any actuarially fair financial lottery. A citizen is able to choose an occupation after having determined the outcome of the financial bet. Since we assume that the functions $u_A(\cdot)$ and $u_F(\cdot)$ are concave, a citizen will not be interested in "pure gambles" where his occupational choice is independent of the outcome of the lottery. Therefore, the only lotteries of interest are those in which his winnings take on two possible values, where one outcome induces the citizen to join the army and the other induces him to stay on the farm. He can choose any lottery with probability π of winning the prize x_A and probability $1-\pi$ of winning prize x_F so long as $\pi x_A+(1-\pi)x_F=0$. If he chooses to join the army whenever the prize is x_A and to farm whenever the prize is x_F, then with probability π the citizen will be in the army and have consumption $c_A=w_A+x_A$ and with probability $1-\pi$ he will be on the farm with consumption $c_F=w_F+x_F$. An equivalent way to say this is that he can choose any probability π and consumptions c_A and c_F such that

$$\pi c_A+(1-\pi)c_F=\pi w_A+(1-\pi)w_F. \tag{1.16}$$

We can now define an equilibrium wage structure for the model of this section.

Definition: Where citizens have identical preferences, a wage structure $(\tilde{w}_A, \tilde{w}_F)$ is a probabilistic recruitment equilibrium if all citizens choose $\pi=\bar{\pi}$ when they are allowed to choose π, c_A, and c_F to maximize expected utility subject to

$$\pi c_A+(1-\pi)c_F=\pi \tilde{w}_A+(1-\pi)\tilde{w}_F.$$

Let us define $V(\pi, w_A, w_F)$ to be the maximum expected utility that a citizen can obtain when the wage structure is (w_A, w_F) if he makes actuarially fair financial bets and joins the army with probability π. Thus $V(\pi, w_A, w_F)$ is the maximum of expected utility function (1.3) subject to the constraint that c_A and c_F satisfy the budget equation (1.16). If the wage structure is (w_A, w_F) and actuarially fair lotteries are available, expected utility is maximized by the following strategy. Choose π to maximize $V(\pi, w_A, w_F)$. For this value of π, choose c_A and c_F to maximize the expected utility subject to (1.16). Make a bet that with probability π pays c_A-w_A and with probability $1-\pi$ pays c_F-w_F. In case of the first outcome, join the army. In case of the second outcome, stay on the farm.

We are able to prove the following result.

Proposition 1.5. Let consumers have identical, continuous von Neumann–Morgenstern utility functions such that $u_A(\cdot)$ and $u_F(\cdot)$ are strictly concave and let there be some feasible wage structure such that everyone prefers being a farmer to being a soldier and another feasible wage structure such that everyone prefers being a soldier to being a farmer. Then there exists a unique wage structure $(\tilde{w}_A, \tilde{w}_F)$ that is a recruitment equilibrium with fair lotteries in consumption goods. In general, this wage structure is not the same as the volunteer army wage structure in the absence of lotteries.

The existence result asserted in Proposition 1.5 is a special case of a result proved in the next section. Here we show by an example that the recruitment equilibrium with fair lotteries may, depending on the variables of the problem, pay soldiers either more or less than the volunteer army wage in the absence of lotteries.

Example 1.3. An equilibrium with financial lotteries. Let all citizens' von Neumann–Morgenstern utility functions take the special form $u_F(c_F) = f(c_F)$ and $u_A(c_A) = f(c_A) - \alpha$ where $f'(\cdot) > 0$ and $f''(\cdot) < 0$. Then the expected utility function expressed in general form by (1.3) becomes

$$U(\pi, c_A, c_F) = \pi(f(c_A) - \alpha) + (1-\pi)f(c_F). \tag{1.17}$$

The special expected utility function (1.17) is maximized subject to (1.16) when

$$c_A = c_F = \pi w_A + (1-\pi)w_F. \tag{1.18}$$

Therefore

$$V(\pi, w_A, w_F) = U(\pi, c_A, c_F) = f(\pi w_A + (1-\pi)w_F) - \pi\alpha. \tag{1.19}$$

Then $V(\pi, w_A, w_F)$ is maximized over π when

$$f'(\pi w_A + (1-\pi)w_F)(w_A - w_F) = \alpha. \tag{1.20}$$

To secure a volunteer army of the right size, the government must offer a feasible wage structure $(\tilde{w}_A, \tilde{w}_F)$ such that (1.20) is satisfied for $\bar{\pi} = \tilde{\pi}$. The fact that $(\tilde{w}_A, \tilde{w}_F)$ must be a feasible wage structure implies that

$$\bar{\pi}\tilde{w}_A + (1-\bar{\pi})\tilde{w}_F = \bar{w}. \tag{1.21}$$

It follows from equations (1.20) and (1.21) that in the presence of actuarially fair lotteries, the wage structure $(\tilde{w}_A, \tilde{w}_F)$ will secure a volunteer army of size $\bar{\pi}$ when

$$f'(\bar{w})(\tilde{w}_A - \tilde{w}_F) = \alpha. \tag{1.22}$$

To get an explicit solution for $\tilde{w}_A$ and $\tilde{w}_F$, we specialize the example further by assuming that $f(c) = \ln c$ for all $c > 0$. Then equation (1.22) implies that

$$\tilde{w}_A - \tilde{w}_F = \bar{w}\alpha, \tag{1.23}$$

and from (1.21) and (1.23) it follows that

$$\tilde{w}_F = (1 - \alpha\bar{\pi})\bar{w} \quad \text{and} \quad \tilde{w}_A = (1 + \alpha - \alpha\bar{\pi})\bar{w}. \tag{1.24}$$

In this example it is straightforward to compare the two wage structures $(\hat{w}_A, \hat{w}_F)$ and $(\tilde{w}_A, \tilde{w}_F)$. With a bit of calculation we can show that when $f(c) = \ln c$,

$$\hat{w}_F = \frac{1}{1 + \bar{\pi}(e^\alpha - 1)}\bar{w} \quad \text{and} \quad \hat{w}_A = \frac{1}{1 + \bar{\pi}(e^\alpha - 1)}\bar{w}e^\alpha. \tag{1.25}$$

Direct computation shows that $\hat{w}_A > \tilde{w}_A$ if $\alpha = 5$ and $\bar{\pi} = 0.1$ whereas $\hat{w}_A < \tilde{w}_A$ if $\alpha = 1$ and $\bar{\pi} = 0.5$. Therefore, we see that, depending on the size of the parameters α and $\bar{\pi}$, $\tilde{w}_A$ can be either larger or smaller than $\hat{w}_A$.

2 Marching to different drums

When people's tastes and abilities differ, it is important that an allocation mechanism selects people for occupations according to principles of comparative advantage. A volunteer army is better able to do this than a draft lottery. But if private lotteries in consumption goods are available, it is typically the case in equilibrium that some citizens will choose to make a gamble in consumption goods and determine their occupation by the outcome of the gamble. In this section, we extend the definition of a probabilistic recruitment equilibrium to the case of a heterogeneous population and we show that this equilibrium is Pareto optimal. To simplify exposition, we will confine our attention here to differences in preferences while assuming that there are no differences in productivity between people and that all income comes from wages.

If there are heterogeneous citizens and no private lotteries, then a volunteer army wage structure $(\hat{w}_A, \hat{w}_F)$ would have the property that there are A citizens i for whom $u_i(\hat{w}_A) \geq u_i(\hat{w}_F)$ and $N - A$ for whom $u_i(\hat{w}_F) \geq u_i(\hat{w}_A)$. Since preferences differ, it will be true that in equilibrium some of the citizens who choose the army will strictly prefer the army at equilibrium wages and some of the citizens who choose to farm will strictly prefer the farm.

Suppose now that actuarially fair lotteries in private goods are available to all consumers. Much as we did in the case of identical consumers, let us define for each i an indirect utility function:

$$V_i(\pi^i, w_A, w_F) = \max_{\pi^i, c_A^i, c_F^i} \pi^i u_A(c_A^i) + (1-\pi^i) u_F(c_F^i)$$
$$\text{subject to } \pi^i c_A^i + (1-\pi^i) c_F^i \le \pi^i w_A + (1-\pi^i) w_F. \tag{2.1}$$

If the wage structure is (w_A, w_F), then $V_i(\pi^i, w_A, w_F)$ is the highest utility level that i can attain by the following type of strategy: Make an actuarially fair bet where with probability π^i the prize is $c_F^i - w_A$ and with probability $1-\pi^i$ the prize is $c_F^i - w_F$, and then choose to join the army if you get the former prize and to farm if you get the latter.

A consumer with access to actuarially fair lotteries when the wage structure is (w_A, w_F) will maximize his expected utility by choosing π^i between 0 and 1 so as to maximize $V_i(\pi^i, w_A, w_F)$. It turns out that under reasonable assumptions, this maximum is unique.

Proposition 2.1. If the functions $u_A^i(\cdot)$ and $u_F^i(\cdot)$ are twice differentiable and strictly concave and if farming is a normal good, then for any wage structure (w_A, w_F) there is one and only one value of π^i that maximizes $V_i(\pi^i, w_A, w_F)$.

Proof: The existence of at least one maximizing π^i is immediate from standard arguments of continuity and boundedness. Taking derivatives and applying the envelope theorem, we find that $\partial^2 V_i(\pi^i, w_A, w_F)/\partial \pi^{i2} < 0$ whenever $\partial V_i(\pi^i, w_A, w_F)/\partial \pi^i = 0$. It is then straightforward from the Kuhn–Tucker conditions that there can be only one value of π^i in the interval $[0,1]$ that maximizes $V_i(\pi^i, w_A, w_F)$. ■

We are therefore entitled to define a *probabilistic supply function* as follows.

Definition. Consumer i's probabilistic supply function for serving in the army is $\pi^i(w_A, w_F)$ where $\pi^i(w_A, w_F)$ is the value of π^i that maximizes $V_i(\pi^i, w_A, w_F)$ on the interval from 0 to 1.

An *allocation* will be fully described by a specification of the probability π^i that each i will be in the army and the contingent consumptions

c_A^i and c_F^i that he will consume if he is in the army and if he farms. We define a *probabilistic recruitment equilibrium wage structure* as one that attracts an army of the right expected size and that makes expected total wages equal to total wealth and a *probabilistic recruitment equilibrium allocation* as an allocation induced by such a wage structure.

Definition. A wage structure $(\tilde{w}_A, \tilde{w}_F)$ is a probabilistic recruitment equilibrium wage structure if $\bar{\pi}\tilde{w}_A + (1-\bar{\pi})\tilde{w}_F = \bar{w}$ and if $1/N \sum_{i=1}^N \pi^i(\tilde{w}_A, \tilde{w}_F) = \bar{\pi}$. An allocation $(\tilde{\pi}^i, \tilde{c}_A^i, \tilde{c}_F^i)$, $i = 1, \ldots, N$, is a probabilistic recruitment equilibrium allocation if $\tilde{\pi}^i = \pi^i(\tilde{w}_A, \tilde{w}_F)$ and if $\tilde{c}_A^i$ and $\tilde{c}_F^i$ maximize $\tilde{\pi}^i u_A^i(c_A^i) + (1-\tilde{\pi}^i) u_F^i(c_F^i)$ subject to $\tilde{\pi}^i c_A^i + (1-\tilde{\pi}^i) c_F^i \leq \tilde{\pi}^i \tilde{w}_A + (1-\tilde{\pi}^i)\tilde{w}_F$.

Proposition 2.2. If all consumers have twice differentiable concave expected utility functions, if farming is a normal good, and if no one would choose an occupation with zero wages if all the bread were paid to the other occupation, then there exists a probabilistic recruitment equilibrium.

Proof: Define the function $\Phi(w_A)$ for $w_A \in [0, \bar{w}/\bar{\pi}]$ so that

$$\Phi(w_A) = \frac{1}{N} \sum_{i=1}^{N} \pi^i \left(w_A, \frac{\bar{w} - \bar{\pi} w_A}{1-\bar{\pi}} \right). \tag{2.2}$$

Our assumptions imply that $\Phi(w_A)$ is a continuous function and that $\Phi(0) = 0$ and $\Phi(\bar{w}/\bar{\pi}) = 1$. From the intermediate value theorem, it follows that as w_A is varied continuously from 0 to $\bar{w}/\bar{\pi}$, $\Phi(w_A)$ takes on all intermediate values between 0 and 1 and in particular for some $\tilde{w}_A \in [0, \bar{w}/\bar{\pi}]$, $\Phi(\tilde{w}_A) = \bar{\pi}$. It is straightforward to verify that $(\tilde{w}_A, \tilde{w}_F)$ is a probabilistic recruitment equilibrium wage where $\tilde{w}_F = (\bar{w} - \bar{\pi}\tilde{w}_A)/(1-\bar{\pi})$. ■

If each individual i joins the army with probability $\pi^i(w_A, w_F)$, then the expected proportion of the population to join the army will be $1/N \sum_{i=1}^N \pi^i(w_A, w_F)$. If the country is large and the lotteries are independent, then with very high probability, the proportion of the population joining the army will be very close to the expected proportion. Likewise, the expected consumption per capita will be very close to $1/N \sum_{i=1}^n [\pi c_A^i + (1-\pi) c_F^i]$. It is therefore reasonable to define a feasible allocation in the following way. [*Remark:* Although the independent lotteries would not take on exactly their expected values, since they will with very high probability come very close in per capita terms, the cost per citizen of insuring this residual risk would be very small and can safely be ignored.]

Definition. An allocation (π^i, c_A^i, c_F^i), $i=1,\dots,N$, is feasible for a country that requires the fraction $\bar{\pi}$ of its population to be in the army and that has a per capita endowment of bread equal to $\bar{w}$ if $\sum_{i=1}^N \pi^i = N\bar{\pi}$ and $1/N\sum_{i=1}^n [\pi c_A^i + (1-\pi)c_F^i] \le \bar{w}$.

Definition. An allocation is Pareto optimal if it is feasible and if there is no other feasible allocation that gives all citizens at least as high an expected utility and at least one citizen a higher expected utility.

Proposition 2.3. A probabilistic recruitment equilibrium allocation is Pareto optimal.

Proof: Let $(\tilde{w}_A, \tilde{w}_F)$ be a probabilistic recruitment equilibrium wage structure and let the corresponding equilibrium allocation be $(\tilde{\pi}^i, \tilde{c}_A^i, \tilde{c}_F^i)$, $i=1,\dots,N$. Then $\sum_{i=1}^N \tilde{\pi}^i = N\bar{\pi}$, and for each i,

$$\pi^i \tilde{c}_A^i + (1-\pi^i)\tilde{c}_F^i \le \pi^i \tilde{w}_A + (1-\pi^i)\tilde{w}_F.$$

Summing these inequalities and dividing by N, we have

$$1/N \sum [\pi^i \tilde{c}_A^i + (1-\pi^i)\tilde{c}_F^i] \le \bar{\pi}\tilde{w}_A + (1-\bar{\pi})\tilde{w}_F \le \bar{w}. \tag{2.3}$$

Therefore the allocation $(\tilde{\pi}^i, \tilde{c}_A^i, \tilde{c}_F^i)$, $i=1,\dots,N$, is feasible.

Suppose that some allocation (π^i, c_A^i, c_F^i), $i=1,\dots,N$, is Pareto superior to $(\tilde{\pi}^i, \tilde{c}_A^i, \tilde{c}_F^i)$, $i=1,\dots,N$. Since by construction $V_i(\tilde{\pi}^i, \tilde{w}_A, \tilde{w}_F) \ge V_i(\pi^i, \tilde{w}_A, \tilde{w}_F)$, it must be that $\pi^i c_A^i + (1-\pi^i)c_F^i \ge \pi^i \tilde{w}_A + (1-\pi^i)\tilde{w}_F$ for all i with strict inequality for some i. Therefore $1/N\sum_{i=1}^N [\pi^i c_A^i + (1-\pi^i)c_F^i] > \pi^i \tilde{w}_A + (1-\pi^i)\tilde{w}_F$. But this means that the allocation (π^i, c_A^i, c_F^i), $i=1,\dots,N$, is not feasible. This proves the theorem. ■

Two examples of probabilistic recruitment equilibrium with heterogeneous tastes

Example 2.1. Here we let utility take the same special functional form as in Example 1.3, but we allow for differences in tastes. In particular, we assume that for each i, $u_F^i(c_F) = \ln c_F$ and $u_A^i(c_A) = \ln c_A - 1/T_i$. The parameter T_i can be interpreted as citizen i's tolerance for the army. Just as in Example 1.3, a consumer with this utility function who can make actuarially fair bets will choose $c_A = c_F$. Then

$$V_i(\pi^i, \tilde{w}_A, \tilde{w}_F) = \ln[\pi^i \tilde{w}_A + (1-\pi^i)\tilde{w}_F] - \frac{\pi^i}{T_i}. \tag{2.4}$$

Let us assume, provisionally, that the value of π that maximizes $V_i(\pi^i, \tilde{w}_A, \tilde{w}_F)$ is an interior solution in the open interval $(0, 1)$. Differentiating with respect to π^i and rearranging terms, we find that the first-order condition for this maximization is equivalent to

$$\pi^i = T_i - \frac{\tilde{w}_F}{\tilde{w}_A - \tilde{w}_F} \tag{2.5}$$

for all i. From (2.5) it follows that

$$\bar{\pi} = \bar{T} - \frac{\tilde{w}_F}{\tilde{w}_A - \tilde{w}_F}, \tag{2.6}$$

where $\bar{T} = 1/N \sum_{i=1}^{N} T_i$. From (2.6) and the feasibility constraint

$$\bar{\pi}\tilde{w}_A + (1-\bar{\pi})\tilde{w}_F = \bar{w},$$

we deduce that

$$\tilde{w}_F = \frac{\bar{T}\bar{w}}{\bar{T}+\bar{\pi}} \quad \text{and} \quad \tilde{w}_A = \frac{(\bar{T}+1)\bar{w}}{\bar{T}+\bar{\pi}}. \tag{2.7}$$

From (2.5) and (2.6), it follows that, for each i,

$$\pi^i = \bar{\pi} + T_i - \bar{T}. \tag{2.8}$$

Therefore, our provisional assumption that π^i is an internal solution for every i is justified if and only if $0 \le \bar{\pi} + T_i - \bar{T} \le 1$ for all i.

The equilibrium contingent consumptions for each i will satisfy

$$\tilde{c}_A^i = \tilde{c}_F^i = \pi^i \tilde{w}_A + (1-\pi^i)\tilde{w}_F = \tilde{w}_F + \pi^i(\tilde{w}_A - \tilde{w}_F). \tag{2.9}$$

From equations (2.5) and (2.9), it follows that

$$\tilde{c}_A^i = \tilde{c}_F^i = T_i(\tilde{w}_A - \tilde{w}_F). \tag{2.10}$$

For this example, let us choose parameter values so that in equilibrium everyone is at an "interior" solution. Specifically, let $\bar{\pi} = 0.25$ and let the individual T_i's be uniformly distributed on the interval $[1, 1.5]$. Then $\bar{T} = 1.25$ and $0 \le \bar{\pi} + T_i - \bar{T} \le 1$ for all i. From equation (2.7), we calculate that

$$\tilde{w}_A = 1.5\bar{w} \quad \text{and} \quad \tilde{w}_F = 0.833\bar{w}. \tag{2.11}$$

Although the equilibrium wages of soldiers are higher than the wages of farmers, each individual will plan contingent consumptions $\tilde{c}_A^i = \tilde{c}_F^i$. But the bigger the risk one takes of losing the preliminary bet and joining the army, the higher will be both of these contingent consumptions. In equilibrium, people with higher tolerance will accept a greater probability of being in the army. In this example, a citizen with tolerance T_i will enter a lottery in which with probability $T_i - 1$ he loses the amount $\tilde{w}_A - \tilde{c}_A^i$ and joins the army and with probability $2 - T_i$ he wins $\tilde{c}_F^i - \tilde{w}_F$ and stays on the farm. In this instance, we find from equation (2.10) that

$$\tilde{c}_A^i = \tilde{c}_F^i = 0.666T_i. \tag{2.12}$$

If there were no private lotteries, then the wage structure would have to be such that the one-fourth of the population with the highest tolerance for the army would choose the army. The marginal individual would be a citizen with tolerance $T_i = \frac{11}{8}$. The voluntary equilibrium wage structure $(\hat{w}_A, \hat{w}_F)$ in the absence of private lotteries would be a feasible wage structure that makes such an individual indifferent between the two occupations. It turns out that

$$\hat{w}_A = 1.63\bar{w} \quad \text{and} \quad \hat{w}_F = 0.789\bar{w}. \tag{2.13}$$

Example 2.2. In this example, preferences will be as in Example 2.1, but we will allow wider variation of the parameter T_i so that in equilibrium some citizens will choose the army with certainty, some will choose to farm with certainty, and some will enter a lottery and make their occupations conditional on the outcome of the lottery. We assume that $\bar{\pi} = 0.25$ and that the T_i's are uniformly distributed over the interval $[1, 4]$.

As in Example 2.1, those citizens who choose π^i in the interior of the interval $[0, 1]$ must satisfy the first-order conditions [eq. (2.5)]

$$\pi^i = T_i - \frac{\tilde{w}_F}{\tilde{w}_A - \tilde{w}_F}.$$

There will be critical tolerance levels T^* and T_* such that in equilibrium all consumers with $T_i \geq T^*$ will choose the army with certainty and all consumers with $T_i \leq T_*$ will farm with certainty. Then T^* and T_* are the upper and lower boundaries of the range of T_i's for which the right side of equation (2.5) takes values between 0 and 1. From equation (2.5), we see that

$$1 = T^* - \frac{\tilde{w}_F}{\tilde{w}_A - \tilde{w}_F} \quad \text{and} \quad T_* = T^* - 1. \tag{2.14}$$

The fraction $\frac{1}{3}(4 - T^*)$ of the population that has $T_i \geq T^*$ will join the army with certainty. The remaining recruits for the army will come from citizens that have $T_* < T_i < T^*$ and whose gambling outcomes result in their joining the army. From equation (2.5), it follows that the expected fraction of the total population that comes from this group to join the army will be

$$\frac{1}{3}\int_{T^*-1}^{T^*}\left(T-\frac{\tilde{w}_F}{\tilde{w}_A-\tilde{w}_F}\right)dT=\frac{1}{3}\left(T^*-0.5-\frac{\tilde{w}_F}{\tilde{w}_A-\tilde{w}_F}\right) \tag{2.15}$$

In equilibrium, the expected fraction of the total population to join the army must be $\bar{\pi}$. Adding the fraction $\frac{1}{3}(4-T^*)$ of the population that will join the army with certainty to the expression (2.15), we find that in equilibrium

$$\bar{\pi}=\frac{1}{3}\left(T^*-0.5-\frac{\tilde{w}_F}{\tilde{w}_A-\tilde{w}_F}+4-T^*\right). \tag{2.16}$$

Using equations (2.14) and (2.16) and substituting in $\bar{\pi}=0.25$, one can show that

$$T^*=3.75 \quad \text{and} \quad \frac{\tilde{w}_F}{\tilde{w}_A-\tilde{w}_F}=2.75. \tag{2.17}$$

Finally, from (2.17) and the constraint $\bar{\pi}\tilde{w}_A+(1-\bar{\pi})\tilde{w}_F=\bar{w}$, it follows that

$$\tilde{w}_A=1.25\bar{w} \quad \text{and} \quad \tilde{w}_F=0.9175\bar{w}. \tag{2.18}$$

In this example, all citizens with $T_i \geq 3.75$ will join the army with certainty and consume $\tilde{w}_A=1.25\bar{w}$. Citizens with $T_i \leq 2.75$ will be certain to stay out of the army and will consume $\tilde{w}_F=0.9175\bar{w}$. A citizen with tolerance level T_i in the interval $[2.75, 3.75]$ will enter a lottery in which with probability $T_i-2.75$ he loses $\tilde{w}_A-\tilde{c}_A^i$ and with probability $3.75-T_i$ he wins $\tilde{c}_F^i-\tilde{w}_F$. If he loses the lottery, he joins the army; if he wins, he farms. In either case, his consumption will be equal to $\frac{1}{3}T_i\bar{w}$.

3 Toward a general theory of occupational choice

Much of what we have said about soldiers and farmers applies as well to garbagemen and bank clerks or to certified public accountants and college professors. The theory generalizes cleanly to the case of many occupational alternatives and to a technology where total output depends in a neoclassical way on the numbers of persons engaged in each occupation. Although the model can be extended without much complication to the level of generality of the Arrow–Debreu model of general equilibrium, to simplify exposition and to emphasize the novel features of our treatment, we will draw our model more starkly.

Let there be N laborers and M occupations where N is "large" relative to M. Each laborer must select exactly one of these occupations. A single consumption good is produced. Total output is determined by a differentiable, concave, linear homogeneous production function, $F(N_1, \dots, N_M)$, where N_i is the number of laborers who select occupation i. Each individual i is assumed to have a (state-dependent) von Neumann–Morgenstern utility function of the form

$$U(\pi^i, c^i) = \sum_{j=1}^{M} \pi_j^i u_j^i(c_j^i), \tag{3.1}$$

where π_j^i is the probability that his occupation will be j and c_j^i is his consumption contingent on the occupation being j.

Let the consumption good be the *numéraire* and let w_j denote the wage rate of occupation j. If laborers have access to all actuarially fair lotteries in consumption goods, then laborer i could obtain any vector of contingent commodities $c^i = (c_1^i, \dots, c_M^i)$ and probability distribution of occupations $\pi^i = (\pi_1^i, \dots, \pi_M^i)$ that satisfies the condition

$$\sum_{j=1}^{M} \pi_j^i c_j^i = \sum_{j=1}^{M} \pi_j^i w_j. \tag{3.2}$$

The way that a laborer can obtain this combination is to participate in a lottery in which for each j there is a probability π_j^i of winning $c_j^i - w_j$. From equation (3.2), we see that this lottery is actuarially fair. If the prize turns out to be $c_j^i - w_j$, then he selects occupation j. The net result of this strategy is that for each j the laborer has the probability π_j^i of selecting occupation j, and his consumption when he collects the wage for that occupation and adds it to his net winnings or losses in the lottery is $w_j + (c_j^i - w_j) = c_j^i$.

With the wage structure $w = (w_1, \dots, w_M)$, the highest expected utility that laborer i can achieve by a strategy of this type with probability distribution π^i is

$$V_i(\pi^i, w) = \max_{c^i} U(\pi^i, c^i) \quad \text{subject to} \quad \sum_{j=1}^{M} \pi_j^i c_j^i \le \sum_{j=1}^{M} \pi_j^i w_j. \tag{3.3}$$

At wage structure w, the laborer will choose π^i to maximize $V_i(\cdot, w)$. If π^i maximizes $V_i(\cdot, w)$ and $V_i(\pi^i, w) = \sum_{j=1}^{M} \pi_j^i u(c_j^i)$, then the strategy in which he enters a lottery that gives him for each occupation j a probability π_j^i of winning $c_j^i - w_j$ is self-enforcing in the sense that if the prize turns out to be $c_j - w_j$ he can do no better than to select occupation j. Suppose he could do better, either by choosing an alternative occupation with certainty or by entering a second actuarially fair lottery and then choosing an occupation. Then the probability distribution π'^i of occupational outcomes and the vector of contingent consumptions c'^i that would obtain after the second stage in the lottery would still satisfy the inequality $\sum_{j=1}^{M} \pi_j'^i c_j'^i \le \sum_{j=1}^{M} \pi_j'^i w_j$. Therefore, since π^i maximizes $V(\cdot, w)$, it follows that $U(\pi^i, c^i) \ge U(\pi'^i, c'^i)$.

In general, there might be more than one choice of π^i that solves the maximization problem (3.3). For each i, let us define the correspondence $\Pi^i(w)$ from the nonnegative orthant of R^M to the subsets of the M-simplex S^M such that

$$\Pi^i(w) = \{\pi^i \in S^M \mid \pi^i \text{ maximizes } V_i(\pi^i, w)\}.$$

We define the correspondence $\Psi(w) = \sum_{i=1}^{N} \Pi^i(w)$. A point $(N_1, \ldots, N_M)$ in $\Psi(w)$ is a distribution of expected numbers of laborers supplied to each occupation at the wage structure w.

Definition. An equilibrium wage structure with lotteries is a vector of wages $\tilde{w}^* = (w_1^*, \ldots, w_M^*)$ such that for some $N^* = (N_1^*, \ldots, N_M^*) \in \Psi(w^*)$ and for all $j = 1, \ldots, M$,

$$w_j^* = \partial F(N^*)/\partial N_j. \tag{3.4}$$

An equilibrium allocation with lotteries is an allocation (c^{*i}, π^{*i}), $i = 1, \ldots, N$, such that π^{*i} maximizes $V_i(\pi^i, w^*)$ on S^M and such that

$$V_i(\pi^{*i}, w^*) = U_i(\pi^{*i}, c^{*i}).$$

To prove the existence of an equilibrium wage structure, we will define a correspondence that satisfies the conditions of the Kakutani fixed point theorem and show that a fixed point for this correspondence is an equilibrium. Let the set $A = \{(N_1, \ldots, N_M) \ge 0 \mid \sum_{i=1}^{M} N_i = N\}$ and let $B = \{(w_1, \ldots, w_M) \ge 0 \mid w_i \le \max_{N \in A} F(N)\}$. Define the correspondence $\Phi(\cdot, \cdot)$ from the convex set $A \times B$ into its subsets so that $\Phi(N, w) = \Psi(w) \times \nabla F(N)$ is the gradient function of $F(N)$. To apply the Kakutani theorem, we must establish that the image sets $\Phi(N, w)$ are convex sets for all (N, w) in $A \times B$. The following result is needed to prove that this is the case.

Proposition 3.1. If the expected utility function $U_i(\pi^i, c^i)$ is concave in c^i for fixed π^i, then the function $V_i(\pi^i, w)$ is a concave function of π^i.

Proof: Consider the probability distributions π and π' in the simplex S^M and let

$$V_i(\pi, w) = \sum_{j=1}^{M} \pi_j u_j^i(c_j^i), \tag{3.5}$$

and

$$V_i(\pi', w) = \sum_{j=1}^{M} \pi'_j u^i_j(c'^i_j), \tag{3.6}$$

where

$$\sum_{j=1}^{M} \pi_j c^i_j = \sum_{j=1}^{M} \pi_j w_j, \tag{3.7}$$

and

$$\sum_{j=1}^{M} \pi'_j c'^i_j = \sum_{j=1}^{M} \pi'_j w_j. \tag{3.8}$$

For λ between 0 and 1, the probability vector

$$\pi(\lambda) = (\pi_1(\lambda), \ldots, \pi_M(\lambda)) = \lambda\pi + (1-\lambda)\pi' \tag{3.9}$$

is also in the simplex S^M. To prove that $V_i(\pi, w)$ is a concave function of π, it suffices to show that $V_i(\pi(\lambda), w) \geq \lambda V_i(\pi, w) + (1-\lambda) V_i(\pi', w)$ for all $\lambda \in [0,1]$.

Multiplying equations (3.7) and (3.8) by λ and $1-\lambda$, respectively, and adding, we obtain

$$\sum_{j=1}^{M} [\lambda\pi_j c^i_j + (1-\lambda)\pi'_j c'^i_j] = \sum_{j=1}^{M} \pi_j(\lambda) w_j. \tag{3.10}$$

For each occupation, $j = 1, \ldots, M$, and for $\lambda \in [0,1]$, let us define

$$\theta_j = \lambda\pi_j / \pi_j(\lambda). \tag{3.11}$$

Then

$$1-\theta_j = (1-\lambda)\pi'_j / \pi_j(\lambda). \tag{3.12}$$

Equation (3.10) can be rewritten as

$$\sum_{j=1}^{M} \pi_j(\lambda)[\theta_j c^i_j + (1-\theta_j) c'^i_j] = \sum_{j=1}^{M} \pi_j(\lambda) w_j. \tag{3.13}$$

It follows from the definition of $V_i(\pi, w)$ that

$$V_i(\pi(\lambda), w) \geq \sum_{j=1}^{M} \pi_j(\lambda) u^i_j(\theta_j c^i_j + (1-\theta_j) c'^i_j). \tag{3.14}$$

But, by assumption, the functions $u^i_j(\cdot)$ are concave functions. Therefore, equation (3.14) implies

$$V_i(\pi(\lambda), w) \geq \sum_{j=1}^{M} \pi_j(\lambda)[\theta_j u^i_j(c^i_j) + (1-\theta_j) u^i_j(c'^i_j)]. \tag{3.15}$$

Using equations (3.11) and (3.12), we see that (3.15) is equivalent to

$$V_i(\pi(\lambda), w) \geq \lambda \sum_{j=1}^{M} \pi_j u_j^i(c_j^i) + (1-\lambda) \sum_{j=1}^{M} \pi_j' u_j^i(c_j'^i). \tag{3.16}$$

Recalling equations (3.1) and (3.2), we see that (3.16) is equivalent to

$$V_i(\pi(\lambda), w) \geq \lambda V_i(\pi, w) + (1-\lambda) V_i(\pi', w). \tag{3.17}$$

Therefore, $V_i(\pi, w)$ is concave in π. ■

With the help of Proposition 3.1, we can prove that Φ has all of the properties needed to apply the Kakutani theorem.

Lemma 3.1. If the expected utility function $U(\pi^i, c^i)$ is continuous and concave in c^i for fixed π, then for each i, the correspondence $\Phi(N, w)$ is an upper semicontinuous correspondence from the closed bounded convex set $A \times B$, and the image sets $\Phi(N, w)$ are convex subsets of $A \times B$ for all (N, w) in $A \times B$.

Proof: It is easily verified that Φ maps the convex set $A \times B$ into its subsets. According to well-known arguments, continuity of preferences and production functions imply that this correspondence is upper semicontinuous. From Proposition 3.1, it follows directly that the image sets of $\Pi^i(w)$ are convex sets for all $w \in A$. Therefore, $\Psi(w) = \sum_{i=1}^{N} \Pi^i(w)$ has convex image sets. The function $\nabla F(N)$ is single valued and hence (trivially) has convex image sets. Therefore, $\Phi(N, w) = (\Psi(w), \nabla F(N))$ must have convex image sets for all $(N, w) \in A \times B$. ■

Now we can prove the existence of equilibrium.

Proposition 3.2. For the model of this section, there exists an equilibrium wage structure with lotteries.

Proof: According to Lemma 3.1, the correspondence Φ satisfies the conditions of the Kakutani fixed point theorem. Therefore, there must exist $N^* \in A$ and $w^* \in B$ such that $N^* \in \Psi(w^*)$ and $w^* = \nabla F(N)$. ■

If equilibrium occurs with large numbers of laborers in each occupation and if the lotteries in which individuals participate are independent, then the proportion of the labor force that selects each occupation will be very close to its expected value. Since the production function has constant returns to scale, per capita output and the marginal products of all factors will be very close to their expected values. It is therefore reasonable to approximate this stochastic economy by a certainty equivalent economy in which feasibility means equality of expected total consumption and aggregate output. (See Remark following Proposition 2.2.)

Definition. A feasible allocation consists of contingent consumptions c_j^i and probabilities π_j^i for all laborers $i = 1, \ldots, N$ and occupations $j = 1, \ldots, M$ such that

$$\sum_{i=1}^{N} \sum_{j=1}^{M} \pi_j^i c_j^i = F(N_1, \ldots, N_M), \tag{3.18}$$

where $N_j = \sum_{i=1}^{N} \pi_j^i$.

When feasibility is defined in this way, an equilibrium wage structure generates a Pareto-optimal allocation.

Proposition 3.3. An equilibrium allocation with lotteries is Pareto optimal.

Proof: We first observe that an equilibrium allocation is feasible. Let (π^{*i}, c^{*i}), $i = 1, \ldots, N$, be an equilibrium allocation. Then for each laborer i,

$$\sum_{j=1}^{M} \pi_j^{*i} c_j^{*i} \leq \sum_{j=1}^{M} \pi_j^{*i} w_j^*. \tag{3.19}$$

Summing equation (3.19) over all i, one has

$$\sum_{i=1}^{N} \sum_{j=1}^{M} \pi_j^i c_j^i \leq \sum_{j=1}^{M} N_j w_j^*, \tag{3.20}$$

where $N_j = \sum_{i=1}^{N} \pi_j^i$. But $w^* = \nabla F(N^*)$. Since $F(N)$ is assumed to be homogeneous of degree 1, it follows from Euler's theorem on homogeneous functions that

$$\sum_{j=1}^{M} N_j w_j^* = F(N^*). \tag{3.21}$$

Therefore,

$$\sum_{i=1}^{N} \sum_{j=1}^{M} \pi_j^i c_j^i \leq F(N^*), \tag{3.22}$$

which establishes that the equilibrium allocation (π^{*i}, c^{*i}), $i = 1, \ldots, N$, is feasible.

Let (π^i, c^i), $i=1,\dots,N$, be an allocation that is Pareto superior to (π^{*i}, c^{*i}), $i=1,\dots,N$. Since $V_i(\pi^{*i}, w^*) \geq V_i(\pi_i, w^*)$, it must be that if $U_i(\pi^i, c^i) > U_i(\pi^{*i}, c^{*i})$, then

$$\sum_{j=1}^{M} \pi_j^i c_j^i > \sum \pi_j^i w_j^*. \tag{3.23}$$

Since preferences are assumed to be strictly monotonic in consumption, it also follows from a familiar argument that if $U_i(\pi^i, c^i) \geq U_i(\pi^{*i}, c^{*i})$, then

$$\sum_{j=1}^{M} \pi_j^i c_j^i > \sum \pi_j^i w_j^*. \tag{3.24}$$

Summing the inequalities (3.23) and (3.24) over all i, we find that

$$\sum_{i=1}^{N} \sum_{j=1}^{M} \pi_j^i c_j^i > \sum_{j=1}^{M} N_j w_j^* = F(N). \tag{3.25}$$

Therefore, the allocation (π^i, c^i), $i=1,\dots,N$, is not feasible. It follows that (π^{*i}, c^{*i}), $i=1,\dots,N$, is Pareto optimal. ■

4 Comments on related literature

Debreu's (1959) discussion of consumers in general equilibrium suggests a way in which occupational choice can be incorporated into a general equilibrium model. He treats a *consumption plan* as a vector the components of which are consumer inputs (with positive signs) and outputs (with negative signs) where "Typically, the inputs of a consumption are various goods and services (related to food, clothing, housing...dated and located); its only outputs are the various kinds of labor performed (dated and located)" (p. 59).

Similarly, Arrow and Hahn (1971) treat labor offered by an individual in different occupations as different commodities. To preserve monotonicity of preferences, they use the convention of treating "time spent not doing work of type x" rather than "time spent doing work of type x" as the commodities, but this is a difference of notation and not of substance.

"Thus, if the individual is capable of teaching for 12 hours a day and also capable of driving a bus for 12 hours a day and if, in fact, he teaches for eight hours a day and does not drive a bus at all, then his demand for 'teaching leisure' is four hours and that for 'bus-driving leisure' is 12 hours" (p. 75).

This modeling strategy is powerful and allows for a natural way of treating most of the issues likely to arise in matters of occupational choice and the allocation of time. But it does introduce a difficulty, both in the interpretation of the major theorems of general equilibrium analysis and in standard applications of comparative statics. The difficulty is that in the model so constructed, it does not seem very reasonable to assume convex preferences. Convexity would demand that a consumer who is indifferent between a certain income with 4 hours of teaching leisure and 12 hours of bus-driving leisure and another income with 12 hours of teaching leisure and 4 hours of bus-driving leisure would be at least as well off with an income halfway between the two incomes and with 8 hours of each kind of leisure. Not only does this assumption seem unappealing on casual grounds but it also seems contrary to the evidence offered by the fact that most people specialize in a single occupation.

Our general model can be thought of as a polar case within the Arrow–Debreu general structure, where instead of assuming that preferences and consumption possibility sets are convex in time spent in each occupation, we assume that each individual must specialize completely. Of course, a more realistic model could be constructed in which there was some convexity and some nonconvexity so that some individuals might in equilibrium choose to allocate their time among several occupations.

The idea of allocating by lottery in the presence of indivisibilities or other nonconvexities is familiar in game theory where equilibrium in mixed strategies plays an essential role (von Neumann and Morgenstern 1944). This idea has also received attention in the literature on the economics of location (see, for example, Mirrlees 1972), the theory of clubs (Hillman and Swan 1983), and the theory of taxation (Stiglitz 1982). Starr (1969) and Arrow and Hahn (1971) demonstrate that in appropriately large economies approximate competitive equilibria will exist even if there are nonconvexities in individual preferences. A careful survey of the recent literature and development of the theme that aggregation smooths can be found in Trockel (1984). The general development of our model of occupational choice is similar in spirit to this work. The extra ingredient of our discussion is that we are able to develop an explicit description of the way in which simple lotteries, with financial prizes only, can be used to achieve the requisite smoothing of aggregate behavior when individuals must make discrete choices.

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