Notes on Indirect Utility

How do we show that the indirect utility function is quasi-convex?

We want to show that if $v(p, m) \geq v(p', m')$, then the indirect utility of the convex combination budget is worse than the indirect utility of the (p, m) budget. That is, for any λ such that $0 < \lambda > 1$,

$$v(\lambda p + (1-\lambda)p', \lambda m + (1-\lambda)m') \leq v(p, m).$$

The key is that anything in the convex combination budget can be afforded with one or the other of the original budgets. Therefore at least one of these two budgets is at least a good as the convex combination budget. So the convex combination is worse than the better one.

How do we show this?

Where $0 < \lambda < 1$, let $p^\lambda = \lambda p + (1-\lambda)p'$ and $m^\lambda = \lambda m + (1-\lambda)m'$. Let $x^\lambda = x(p^\lambda, m^\lambda)$. Then $p^\lambda x^\lambda \leq m^\lambda$ (since $x^\lambda = x(p^\lambda, m^\lambda)$ is something you can afford with income m^λ.) But this means that $\lambda p x^\lambda + (1-\lambda)p' x^\lambda \leq \lambda m + (1-\lambda)m'$. This implies that $\lambda(m - px) + (1-\lambda)(m' - p'x) \geq 0$, which implies that either $m \geq px^\lambda$ or $m' \geq p'x^\lambda$, (or possibly both) which implies that at least one of the following two things are true: $px^\lambda \leq m$ and hence $v(p^\lambda, m^\lambda) \leq v(p, m)$ or $p'x^\lambda \leq m'$ and hence $v(p^\lambda, m^\lambda) \leq v(p', m')$. So it must be that $v(p^\lambda, m^\lambda) \leq \max\{v(p, m), v(p', m')\}$.

Proof of Roy's Identity

Roy's identity is as follows:

$$x_j(p, m) = -\frac{\partial v(p, m)}{\partial p_j} \div \frac{\partial v(p, m)}{\partial m)}. \tag{1}$$

By definition, $v(p, m) = u(x(p, m))$. Where $u_i(x)$ denotes the partial derivative of u with respect to its ith argument, the first order conditions for maximization tell us that for some $\lambda(p, m) > 0$ and all $i = 1, \ldots n$,

$$u_i(x(p, m)) = \lambda(p, m)p_i. \tag{2}$$

We first show that

$$\frac{\partial v(p, m)}{\partial m} = \lambda(p, m). \tag{3}$$

We know that

$$\frac{\partial v(p, m)}{\partial m} = \sum_i u_i(x(p, m))\frac{dx_i(p, m)}{dm} = \lambda \sum_i p_i \frac{dx_i(p, m)}{dm}. \tag{4}$$

We also know that for all m, $\sum_i p_i x_i(p, m) = m$. Therefore it must be that

$$\sum_i p_i \frac{dx_i(p, m)}{dm} = 1.$$

It follows that

$$\frac{\partial v(p,m)}{\partial m} = \lambda \sum_i p_i \frac{dx_i(p,m)}{dm} = \lambda. \tag{5}$$

Now let us differentiate with respect to p_j.

$$\frac{\partial v(p,m)}{\partial p_j} = \sum_i u_i\left(x(p,m)\right)\frac{dx_i(p,m)}{dp_j} = \lambda \sum_i p_i \frac{dx_i(p,m)}{dp_j}. \tag{6}$$

Differentiate both sides of the budget equation with respect to p_j to see that

$$\sum_i p_i \frac{dx_i(p,m)}{dp_j} + x_j(p,m) = 0.$$

Therefore

$$\sum_i p_i \frac{dx_i(p,m)}{dp_j} = -x_j(p,m). \tag{7}$$

Substitute Equation 7 into Equation 6 to find that

$$\frac{\partial v(p,m)}{\partial p_j} = -\lambda x_j(p,m).$$

Recalling Equation 5, we then have Equation 1, which is Roy's Identity.