Final Exam Economic 210A, Fall 2009

Answer any 7 questions.

For a person with income m, let us define the *compensating variation* of a price change from price vector p to price vector p' to be the amount of additional income (positive or negative) that the person would have to be given to make him exactly as well off after the price change as before. Thus compensating variation is the solution CV to the equation $v(p', m + CV) = v(p, m)$ where v is the indirect utility function. Let us define *equivalent variation* to be the amount that someone who currently has income m and prices p would be willing to pay in order to avoid a price change such that the new price vector is p' and her income is m. Thus equivalent variation is the solution to the equation $v(p, m - EV) = v(p', m)$. In general, compensating variation and equivalent variation are not the same, but when preferences are quasi-linear, they are the same.

Question 1. Consider a person who consumes two commodities x and y and has utility function

$$u(x, y) = x + y - \frac{1}{2}y^2.$$

Let good x be the numeraire and consider price vectors of the form $p = (1, p_y)$ where p_y is the price of good y.

a. For what price-income combinations does this consumer choose to consume positive amounts of both goods. For price income combinations such that he consumes positive amounts of each good, write an equation for this person's Marshallian demand function for each good.

ANSWER He consumes a positive amount of good 1 if $p_y \leq 1$ and a positive amount of good 2 if $m > p_y(1 - p_y)$. If he consumes positive amounts of both goods, demand for y is $1 - p_y$ and demand for x is $m - p_y(1 - p_y)$.

b. Let $v(1, p_y, m)$ be this person's indirect utility function at price vector $(1, p_y)$ and income m. Write an equation for $v(1, p_y, m)$ that applies at all price-income situations such that he chooses some of each good. Write your equation in as simple a form as possible.

ANSWER $v(1, p_y, m) = m + \frac{1}{2}(1 - p_y)^2$.

c. Suppose that this consumer initially consumes positive amounts of both goods at income m and prices $(1, p_y)$. The prices change to $(1, p'_y)$. Solve for the compensating variation of this price change. Solve for the equivalent variation of this price change. Show that compensating and equivalent variation are equal.

ANSWER

$$CV = \frac{1}{2}\left((1 - p_y)^2 - (1 - p'_y)^2\right).$$

$$EV = \frac{1}{2}\left((1 - p_y)^2 - (1 - p'_y)^2\right).$$

d. If $p_y = 1/2$, $p'_y = 1/4$, and $m = 2$, solve for the compensating variation of a change in prices from $(1, p_y)$ to $(1, p'_y)$.

ANSWER -5/32

Question 2. Consider a person who consumes two commodities x and y and has utility function

$$u(x, y) = x^{1/2}y^{1/2}.$$

Let good x be the numeraire and consider price vectors of the form $p = (1, p_y)$ where p_y is the price of good y.

a. Write down this person's Marshallian demand function for each of the two goods.

ANSWER Marshallian demand functions $x(1, p_y, m) = \frac{m}{2}$, $y(1, p_y, m) = \frac{m}{2p_y}$.

b. Let $v(1, p_y, m)$ be this person's indirect utility function at price vector $(1, p_y)$ and income m. Write an equation for $v(1, p_y, m)$ that applies at all price-income situations such that he chooses some of each good.

ANSWER

$$v(1, p_y, m) = \frac{m}{2}p_y^{-1/2}.$$

c. Suppose that $m = 100$, $p_y = 1$, and $p'_y = 4$. Calculate the compensating variation of a change in the price vector from $(1, p_y)$ to $(1, p'_y)$.

ANSWER CV=100

d. Suppose that $m = 100$, $p_y = 1$, and $p'_y = 4$. Calculate the equivalent variation of a change in the price vector from $(1, p_y)$ to $(1, p'_y)$.

ANSWER EV=50

Question 3. A firm has the production function

$$u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{s/\rho}$$

where $\rho = -1$ and where $0 < s < 1$.

a..Is this production function homogeneous? If so, of what degree is it homogeneous? Does it have positive marginal products? Is it a concave function? (Prove your answers.)

ANSWER It is homogeneous of degree s. It has positive marginal products. It is a concave function. One way to prove that it is a concave function is to check the Hessian conditions. This is a bit cumbersome, but not too hard to show. Another way is as follow. First define $g(x_1, x_2) = \left(x_1^{-1} + x_2^{-1}\right)^{-1}$. Then $g(x_1, x_2)$ is a is a convex function. You can check this by looking at the Hessian. Since $g(x_1, x_2)$ is convex, it is also quasi-convex. Note that $u(x_1, x_2) = g(x_1, x_2)^{-s}$

where $1 > s > 0$. Thus u is a strictly decreasing monotone transformation of g. A strictly decreasing function of a quasi-convex function is a quasi-concave function. (Proof of this is easy.) We have the result that a quasi-concave function that is homogeneous of degree less than 1 is concave. Since u is homogenous of degree s and quasi-concave, it is concave.

b.. Find the conditional factor demand functions for factors 1 and 2 with factor prices w_1 and w_2 and output y.

ANSWER

$$x_1(w_1, w_2, y) = y^{1/s} \left(\frac{w_1^{1/2} + w_2^{1/2}}{w_1^{1/2}} \right).$$

$$x_2(w_1, w_2, y) = y^{1/s} \left(\frac{w_1^{1/2} + w_2^{1/2}}{w_2^{1/2}} \right).$$

c. Find the cost function $c(w_1, w_2, 1)$ for producing 1 unit of output. Find the cost function $c(w_1, w_2, y)$ of producing y units.

ANSWER

$$c(w_1, w_2, 1) = \left(\sqrt{w_1} + \sqrt{w_2}\right)^2.$$

$$c(w_1, w_2, y) = y^{1/s} \left(\sqrt{w_1} + \sqrt{w_2}\right)^2.$$

d. Suppose that $s = 1/2$, $w_1 = 4$ and $w_2 = 1$. If the firm can sell its output at a competitive price of \$72 per unit, how many units should it produce to maximize its profits?

ANSWER If output is y, profits are $72y - c(4, 1, y) = 72y - 9y^2$. The are maximized when $72 = 18y$ and hence when $y = 4$.

Question 4. Consider the function $f(x, y) = 2xy - x^2 - y^2$.

a. What is the gradient of f at the point $(x, y) = (-2, 1)$.

ANSWER $(6, -6)$

b.At the point $(x, y) = (-2, 1)$, what is the directional derivative of f in the direction $(3/5, 4/5)$?

ANSWER $-6/5$

c. If we consider a surface with equation $z = 2xy - x^2 - y^2$ where z is the altitude at the point with coordinates (x, y), and if an ant is currently located at a point where $x = -2$ and $y = 1$, in what (x, y) direction should it move in order to ascend most steeply? If it moves in this direction, what is the derivative of its altitude with respect to the distance moved in the (x, y) plane?

ANSWER At this point, the gradient is $(-6, 6)$. The ant should move in the direction of the gradient. The vector of one unit length pointed in this direction is $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. The derivative of the ant's elevation with respect to distance moved in the (x, y) plane is the dot product is the inner product of these two vectors, which is $6\sqrt{2}$.

Question 5. A consumer has utility function

$$u(x_1, x_2, \ldots, x_n) = x_1 + \sum_{i=2}^{n} a_i \ln(x_i + 1)$$

where $a_i > 0$ for all i. We will let good 1 be the numeraire, so that throughout the remainder of this question you only need to consider price vectors p of the form $p = (1, p_2, \ldots, p_n)$.

a. At price-income combinations such that this consumer consumes positive amounts of every good, write down this consumer's Marshallian demand function for good i for $i \geq 2$. Write an expression for this consumer's Marshallian demand function for good 1 where good 1 is the numeraire.

ANSWER The first order condition is that $p_i = \frac{a_i}{1+x_i}$ for each $i \geq 2$. That implies $x_i = \frac{a_i}{p_i} - 1$. Demand for good 1 is

$$m - \sum_{2}^{n} p_i x_i = m - \sum_{2}^{n} (a_i - p_i).$$

b. For prices and incomes such that the consumer buys positive amounts of every good, write down this consumer's indirect utility function. Is this function of the Gorman polar form? Explain.

ANSWER The indirect utility function is

$$m - \sum_{2}^{n} a_i + \sum_{2}^{n} p_i + \sum_{2}^{n} a_i \ln\left(\frac{a_i}{p_i}\right).$$

This function is of the Gorman polar form $v_i(p, m) = F(p)m + G(p)$ where $F(p) = 1$ and

$$G(p) = -\sum_{2}^{n} a_i + \sum_{2}^{n} p_i + \sum_{2}^{n} a_i \ln\left(\frac{a_i}{p_i}\right)$$

c. For prices and incomes such that the consumer buys positive amounts of every good, write down this consumer's expenditure function and verify that Shepherd's lemma applies.

ANSWER

$$E(p, u) = u + \sum_{2}^{n} a_i - \sum_{2}^{n} p_i - \sum_{2}^{n} a_i \ln\left(\frac{a_i}{p_i}\right)$$

For $j = 2, \ldots, n$, the partial derivative of E with respect to p_j is $-1 + a_j/p_j$. We have shown that $x_j = a_j/p_j - 1$. This is as it should be, given Shepherd's lemma.

d. What has to be true of prices and income if a consumer buys positive amounts of every good?

ANSWER We need $p_j \leq a_j$ for all j and we need $m > \sum_2^n (a_j - p_j)$.

Question 6. Mr. O.B. Kandle will live for 3 more years. He is retired and has a wealth of W. His only source of income is return on the retirement account that he has entrusted to his good friend, Bernie Madoff. Madoff assures him that he will make a rate of return of twenty-five percent per year on any money that he leaves in his retirement account. Mr. Kandle's utility function is

$$U(x_1, x_2, x_3) = \ln x_1 + a \ln x_2 + b \ln x_3$$

where x_i is the amount of money that he spends on consumption in year i. Mr. Kandle is absolutely certain that Madoff will make good on his promises. Assume that he must choose his lifetime savings and consumption plan today and must stick to this plan.

a. Write down Mr. Kandle's intertemporal budget constraint.

ANSWER

$$x_1 + \frac{1}{1.25} x_2 + \frac{1}{(1.25)^2} x^3 = W.$$

b. Suppose that $a = b = 1$. Find the consumption path (x_1, x_2, x_3) that Mr. Kandle should choose to maximize his utility subject to his budget.

ANSWER Setting marginal rates of substitution equal to price ratios, we see that $x_2 = (5/4)x_1$ and $x_3 = (25/16)x_1$. Substituting into the budget constraint, we have $x_1 = W/3$. Then we must also have $x_2 = (5/4)x_1 = 5W/12$, $x_3 = (25/16)x_1 = 25W/48$.

c. Suppose that $a = 4/5$ and $b = 16/25$. Find the consumption path that Mr. Kandle should choose to maximize his utility subject to his budget.

ANSWER Setting marginal rates of substitution equal to price ratios, we must have $x_1 = x_2 = x_3$. Substituting into the budget constraint, we must have $x_1(1 + 4/5 + 16/25) = W$ and hence $x_1 = x_2 = x_3 = 25/61$.

d. Suppose that $a = 4/5$ and $b = 4/5$. Find the consumption path that Mr. Kandle should choose to maximize his utility subject to his budget.

ANSWER $x_1 = x_2 = 4W/13$, $x_3 = 25W/52$.

Question 7. Tiger consumes two goods, x and y and his utility function is

$$U(x, y) = \min\{\sqrt{xy}, x\}.$$

a. Draw a graph, showing a few of Tiger's indifference curves. Does Tiger have convex preferences?

ANSWER On your graph, draw the diagonal line $x = y$. The indifference curve running through a point, $(\bar{x}, \bar{x})$ on this line extends vertically above this point and follows the locus of points on the curve $\sqrt{xy} = \bar{x}$ below your diagonal line. Tiger has convex preferences.

b. What is Tiger's Marshallian demand function for each of the two goods?

ANSWER $x(p_x, p_y, m) = y(p_x, p_y, m) = m/(p_x + p_y)$ if $p_x \geq p_y$.
$x(p_x, p_y,) = m/(2p_x)$ and $y(p_x, p_y, m) = m/2p_y$ if $p_x \leq p_y$.

Question 8. Willy owns a factory on the banks of a river that occasionally floods. He has no other assets. If there is no flood this spring, Willy's factory will be worth \$500,000. If there is a flood, the factory will be worthless. Willy is an expected utility maximizer with von Neumann Morgenstern utility function $u(w) = \ln w$ where w is his wealth. Willy believes that the probability of a flood is 1/10. Willy is offered a chance to buy as much flood insurance as he likes at a cost of \$$c$ per dollar's worth of insurance. The way this policy works is that if he buys \$$X$ worth of flood insurance and if there is no flood, he must pay a total of \$$cX$ in insurance premiums. If there is a flood, he doesn't have to pay his insurance premium, and he receives a payment of \$$X$ from the insurance company.

a. Write down Willy's budget constraint for the contingent commodities "wealth if no flood" and "wealth if flood".

ANSWER If he buys x units of insurance, his wealth will be $W_F = x$ if there is a flood and $W_N = 500,000 - cx = 500,000 - cW_F$ if there is no flood. Rearranging terms, we can write the budget as

$$W_N + cW_F = 500,000.$$

b. At what price c, will Willy buy just enough insurance so that his wealth is the same, whether or not there is a flood?

ANSWER $c = 1/9$.

c. Write down a formula for the amount of insurance that Willy will buy as a function of the cost \$$c$ per dollar of insurance. What is the price elasticity of Willy's demand for insurance?

ANSWER He will choose $w_F = 50,000/c$ and $w_n = 45,000$. The amount of insurance that he buys is $w_F = 50,000/c$. His price elasticity of demand for insurance is -1.