Some logic notation

- $A \vee B$ means A is true or B is true (possibly both are true).
- $A \wedge B$ means A and B are both true.
- $\neg A$ means A is not true.
- $A \Rightarrow B$ means A implies B.
- $A \Leftrightarrow B$ means A implies B and B implies A.
- $\exists_{x \in X}$ means There exists an element x in the set X
- $\forall_{x \in X}$ means for every element x in the set X.

Proving that $\succ$ is transitive if $\succeq$ is transitive.

- Assume that $\succeq$ is transitive on X. That is: $\forall_{x,y,z \in X}$, $(x \succeq y) \wedge (y \succeq z) \Rightarrow x \succeq z$.
- Define $x \succ y \Leftrightarrow (x \succeq y) \wedge \neg(y \succeq x)$.
- We would like to show that $\forall_{x,y,z \in X}$, $(x \succ y) \wedge (y \succ z) \Rightarrow x \succ z$. .
- Suppose that $(x \succ y) \wedge (y \succ z)$. From the definition of $\succ$, it follows that $(x \succeq y) \wedge (y \succeq z)$. Then transitivity of $\succeq$ implies that $x \succeq z$.
- To show that $x \succ z$, we also need to show that $(x \succ y) \wedge (y \succ z) \Rightarrow \neg(z \succeq x)$.

Lets try showing this by contradiction.

- The statement that $(x \succ y) \wedge (y \succ z) \Rightarrow \neg(z \succeq x)$ would be contradicted if and only if $\exists_{x,y,z \in X}$ such that $(x \succ y) \wedge (y \succ z) \wedge (z \succeq x)$.
- But if $(x \succ y) \wedge (y \succ z) \wedge (z \succeq x)$, then, since the definition of $\succ$ implies that $x \succeq y$, we have $z \succeq x$ and $x \succeq y$.
- Therefore by transitivity, we must have $z \succeq y$.
- But we have assumed that $y \succ z$, which by definition implies $\neg(z \succeq y)$.
- Thus we have shown that if $(x \succ y) \wedge (y \succ z) \wedge (z \succeq x)$, then it must be that $z \succeq y$ and $\neg(z \succeq y)$, which is a contradiction.

Closing the Deal

- Since the assumption that $(x \succ y) \wedge (y \succ z) \wedge (z \succeq x)$ leads to a contradiction, it must be that $(x \succ y) \wedge (y \succ z) \Rightarrow \neg(z \succeq x)$.
- Previously we showed that $(x \succ y) \wedge (y \succ z) \Rightarrow x \succeq z$, so we now know that $(x \succ y) \wedge (y \succ z) \Rightarrow (x \succeq z) \wedge \neg(z \succeq x)$, which, given the from the definition of $x \succ z$, is equivalent to $(x \succ y) \wedge (y \succ z) \Rightarrow (x \succ z)$.
- This proves transitivity of $\succ$.

Negative Transitivity

Consider a binary relation P on X that has these two properties. We will think of this as strict preference.

- P is asymmetric. This means that $xPy \Rightarrow \neg(yPx)$.
- P has negative transitivity on X. This means for any x, y, and z in X. $xPy \rightarrow (xPz) \vee (zPy)$. (Stated another way, if one thing is preferred to the other , then for any z,either z is worse than the better one or better than the worse one.)
- Let us define $\succeq$ as follows $x \succeq y \Leftrightarrow \neg(yPx)$. (We can think of $x \succeq y$ as meaning y is not preferred to x.)

What $\succeq$ inherits from P.

- If P is asymmetric and negatively transitive, then $\succeq$ defined above is complete and transitive.
- Hint: (transitivity) Write down the contrapositive of $xPy \rightarrow (xPz) \vee (zPy)$. Translate the result in terms of $\succeq$. Show that the result implies transitivity of $\succeq$.
- Hint: (completeness). Suppose that $\neg(x \succeq y)$. What does that say in terms of P? Use the fact that P is asymmetric to show that that if $\neg(x \succeq y)$ then it must be that $y \succeq x$. Voila!
- The two items above are just hints. You need to supply the details that prove each of these claims.