Name ______________________________

Midterm Examination: Economics 210A

November 6, 2013

Answer Question 1 and any 3 of the remaining 4 questions. Good luck.

1) A consumer has utility function

$$u(x_1, x_2) = \frac{1}{\frac{1}{x_1} + \frac{4}{x_2}}.$$

A) Find this consumer's Marshallian demand function for each good.

Answer:

$$x_1(p_1, p_2, m) = \frac{m}{p_1 + 2p_1^{1/2}p_2^{1/2}}$$

$$x_2(p_1, p_2, m) = \frac{2m}{2p_2 + p_1^{1/2}p_2^{1/2}}$$

B) Find this consumer's indirect utility function.

Answer: You can get your answer by substituting the Marshallian demands that you found into the definition of indirect utility which is

$$v(p, m) = u(x_1(p, m), x_2(p, m)).$$

$$v(p_1, p_2, m) = \frac{m}{p_1 + 4p_1^{1/2}p_2^{1/2} + 4p_2}$$

Another way of writing the same answer is

$$v(p_1, p_2, m) = \frac{m}{\left(p_1^{1/2} + 2p_2^{1/2}\right)^2}$$

C) Find this consumer's expenditure function.

Since $v(p, e(p, u)) = u$, it follows that

$$\frac{e(p_1, p_2, u)}{p_1 + 4p_1^{1/2}p_2^{1/2} + 4p_2} = u$$

and therefore

$$e(p_1, p_2, u) = u\left(p_1 + 4p_1^{1/2}p_2^{1/2} + 4p_2\right).$$

A side remark: *To check your answer (and see how these structures are related), you could notice that the denominator of the indirect utility function is* *a CES function of the p's. You might recall from our classroom discussion that if a utility function is of the form*

$$(c_1 x_1^a + c_2 x_2^a)^{1/a}$$

then the corresponding expenditure function is of the form

$$(c_1' p_1^b + c_2' p_2^b)^{1/b} u.$$

When this is the case, the elasticity of substitution of the direct utility function is $1/(1-a)$ *and the elasticity of substitution of the indirect utility function, which is* $1/(1-b)$ *must be inverses of each other. That is,* $1/(1-b) = 1-a$. *Solving this equation for b, we have* $b = -a/(1-a)$. *In our example,* $a = -1$, *so* $b = 1/2$. *Thus the expenditure function is of the form*

$$e(p_1, p_2, u) = u(c_1' p_1^{1/2} + c_2' p_2^{1/2})^2 u.$$

We also showed that $c_i' = c_i^\sigma = c_i^{1/2}$, *So we would have*

$$e(p_1, p_2, u) = \left(p_1^{1/2} + 2p_2^{1/2}\right)^2 u = \left(p_1 + 4p_1^{1/2} p_2^{1/2} + 4p_2\right) u.$$

D) Find this consumer's Hicksian demand function for each good.

The Hicksian demands are equal to the corresponding partial derivatives of the expenditure function.

We find

$$h_1(p_1, p_2, u) = u(1 + 2p_1^{-1/2} p_2^{1/2})$$

and

$$h_2(p_1, p_2, u) = u(4 + 2p_1^{1/2} p_2^{-1/2}).$$

E) Find the income and substitution effects of a change in the price of good 1 on the demand for good 1.

The substitution effect is the partial derivative of $h_1(p_1, p_2, m)$ *with respect to* p_1. *That is*

$$-u p_2^{1/2} p_1^{-3/2}.$$

We may want to express this substitution effect entirely as a function of prices and income. Thus we would make use of the fact that $u = v(p_1, p_2, m)$. *Then we have the substitution effect is*

$$\frac{-m p_2^{1/2} p_1^{-3/2}}{p_1 + 4p_1^{1/2} p_2^{1/2} + 4p_2}$$

The income effect is

$$-x_1(p_1, p_2) \frac{\partial x(p_1, p_2, m)}{\partial m} = -\frac{m}{\left(p_1 + 2p_1^{1/2} p_2^{1/2}\right)^2}$$

2) A consumer has utility function

$$u(x_1,x_2,x_3)=\frac{1}{\frac{1}{x_1}+\frac{4}{x_2}}+2\sqrt{x_3}.$$

A) If this consumer has income m and the prices of the three goods are p_1, p_2, and p_3, how much of her income will she spend on good 1. (Hint: make use of part B of your answer to the previous question.)

Answer: If the consumer spends a total of s on goods 3 then she will get s/p_3 units of good 3 and will have $m-s$ left to spend on goods 1 and 2. Her utility will then be

$$2\sqrt{s/p_3}+v(p_1,p_2,m-s)=2\sqrt{s/p_3}+\frac{m-s}{\left(p_1^{1/2}+2p_2^{1/2}\right)^2}.$$

She will choose s to maximize this. At an interior maximum, the derivative of this expression with respect to s must be zero. Then

$$(sp_3)^{1/2}-\left(p_1^{1/2}+2p_2^{1/2}\right)^2=0$$

and so

$$s=\frac{\left(p_1^{1/2}+2p_2^{1/2}\right)^4}{p_3}. \tag{1}$$

and total expenditure on goods 1 and 2 is

$$m-s=m-\frac{\left(p_1^{1/2}+2p_2^{1/2}\right)^4}{p_3} \tag{2}$$

B) Write down the consumer's Marshallian demand function for each good. (Hint: make use of your answer to parts A of this question and the previous question.)

Answer: The consumer will choose to allocate the expenditure $m-s$ between goods 1 and 2 in such a way as to maximize utility subject to $p_1x_2+p_2x_2=m-s$. So we can just apply the Marshallian demand functions from Problem 1 with income $m-s$ These are:

$$x_1(p_1,p_2,m)=\frac{m-s}{p_1+2p_1^{1/2}p_2^{1/2}}=\left(m-\frac{\left(p_1^{1/2}+2p_2^{1/2}\right)^4}{p_3}\right)\left(\frac{1}{p_1+2p_1^{1/2}p_2^{1/2}}\right)$$

$$x_2(p_1,p_2,m)=\frac{2(m-s)}{2p_2+p_1^{1/2}p_2^{1/2}}=\left(m-\frac{\left(p_1^{1/2}+2p_2^{1/2}\right)^4}{p_3}\right)\left(\frac{2}{2p_2+p_1^{1/2}p_2^{1/2}}\right)$$

An excellent answer would also note that there will be an interior solution if and only if

$$\left(m - \frac{\left(p_1^{1/2} + 2p_2^{1/2}\right)^4}{p_3}\right) > 0.$$

If this inequality is reversed, there will be a corner solution in which the consumer consumes only good 3.

3) Consider the function

$$f(x_1, x_2, x_3) = \ln\left(x_1^2 + x_2^2 + x_3^2\right).$$

Explain your answers do each of the following questions:
A) Is the function f homogeneous? If so, of what degree?

Answer: No. $f(tx) = \ln t^2 f(x) = 2\ln t + f(x)$. *A function is homogeneous if and only if* $f(tx) = t^k f(x)$ *for all* t *and some* k. *There is no value of* k *such that* $2\ln t + f(x) = t^k f(x)$.
B) Is the function f homothetic?

Answer: Yes. $f(x)$ *is a strictly increasing function of the homogeneous function* $x_1^2 + x_2^2 + x_3^2$.

C) Find the gradient vector of f at the point $(x_1, x_2, x_3) = (1, 3, 5)$.

The gradient is $(2/35, 6/35, 10/35)$.
D) At the point $(1, 3, 5)$, find the directional derivative of f in the direction $(2/9, -2/9, 1/9)$.

Answer: For the directional derivative in the direction of a vector x*, we need to take the inner product of the gradient with the normalized vector* $x/|x|$ *where* $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. *In our case,* $x/|x| = (2/3, -2/3, 1/3)$
The directional derivative is

$$\frac{1}{3 \times 35}(2 \times 2 - 6 \times 2 + 10 \times 1) = \frac{2}{105}.$$

2) Ada and Bo both have have convex preferences defined on a set X, but their preferences are not identical.

A) Let x be a commodity bundle in X. Let A be the set of bundles that both Ada and Bob like better than x. Is A necessarily a convex set? If so, prove it. If not, show a counterexample.

Answer: If Ada and Bob both have convex preferences, then the sets A *and* B *are both context sets. The set of bundles that both like better than* x *is the set* $A \cap B$. *The intersection of two convex sets is a convex set. (A good answer would also prove this last claim.)*

B) Let x be a commodity bundle in X. Let C be the set of bundles such that if $z \in C$, then $z = y_A + y_B$ for some $y_A \in X$ and $y_B \in X$, $z = y_A + y_B$ such that Ada prefers y_A to x and Bo prefers y_B to x. Is C necessarily a convex set. If so, prove it. If not, show a counterexample.

Answer: This is true. Let z and z' be two points in C. Then $z = y_A + y_B$ for some $y_A \in A$ and some $y_B \in B$ and $z' = y'_A + y'_B$ for some $y'_A \in A$ and some $y'_B \in B$. Now for any λ between 0 and 1, and any two points z and z' in C, let $z(\lambda) = \lambda z + (1-\lambda)z'$. Then

$$z(\lambda) = (\lambda y_A + (1-\lambda)y'_A) + (\lambda y_B + (1-\lambda y'_B)).$$

Since A and B are convex sets, it must be that $\lambda y_A + (1-\lambda)y'_A \in A$ and $\lambda y_B + (1-\lambda y'_B) \in B$. Since

$$z(\lambda) = (\lambda y_A + (1-\lambda)y'_A) + (\lambda y_B + (1-\lambda y'_B)),$$

it must be that $z(\lambda) \in C$ and hence C is a convex set.

5) Will's preferences are represented by the utility function

$$u(x_1, x_2) = \min\{x_1^{1/2}x_2^{1/2}, x_2\}.$$

A) Draw a couple of indifference curves for Will.

Answer: Let good 1 be on the horizontal axis and good 2 on the vertical axis. To draw a typical indifference curve for Will, take a point (x, x) on the diagonal. The indifference curve through this point includes a straight horizontal line extending to the right of true point (x, x) and above and to the left of (x, x) is the locus of points (x_1, x_2) satisfying the equation $x_1 x_2 = x$.

B) Find Will's Marshallian demand functions for goods 1 and 2.

Answer:
If $p_1 \le p_2$, $x_1(p_1, p_2) = x_2(p_1, p_2) = m/(p_1 + p_2)$.
If $p_1 > p_2$, then $x_1 = m/(2p_1)$ and $x_2 = m/(2p_2)$.