

Midterm Examination: Economics 210A

October 2011

The exam has 6 questions. Answer as many as you can. Good luck.

1) A) Must every quasi-concave function must be concave? If so, prove it. If not, provide a counterexample. (In all answers where you provide a counterexample, you must show that your example is really a counterexample.)

Answer: Not every quasi-concave function is concave. Here is a counterexample. Define the function F with domain \mathfrak{R}^+ (the positive real numbers.) such that $F(x) = x^2$. We show two things: 1) This function is quasi-concave.¹ To see this, note that F is a strictly increasing function on \mathfrak{R}^+ . Therefore if $F(y) \geq F(x)$, it must be that $y \geq x$ and hence for any $t \in [0, 1]$, $ty + (1 - t)x \geq x$. Since F is an increasing function, it follows that $F(ty + (1 - t)x) \geq F(x)$. Therefore F is quasi-concave. 2) The function F is not concave. To see this, note that $F(2) = 4$ and $F(0) = 0$, but

$$F\left(\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0\right) = F(1) = 1 \leq \frac{1}{2}F(2) + \frac{1}{2}F(0) = 2.$$

This cannot be the case if F is a concave function.

B) Must every concave function be quasi-concave? If so, prove it. If not, provide a counterexample.

Answer: Every concave function is quasi-concave. Proof. If f is concave, its domain is a convex set A . For all x and y in A , and t between 0 and 1, if

$$f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y). \quad (1)$$

From the Expression 1 it follows that

$$f(tx + (1 - t)y) \geq f(y) + t(f(x) - f(y)). \quad (2)$$

If $f(x) \geq f(y)$, then it follows from Expression 2 that

$$f(tx + (1 - t)y) \geq f(y). \quad (3)$$

which means that f is quasi-concave.

¹By the way, $F(x) = x^2$ defined on the entire real line would not be quasi-concave. Can you show this?

2) Let F and G be real-valued concave functions with the same domain, A . Define the function H so that for all $x \in A$, $H(x) = F(x) + G(x)$. Is H a concave function? If so, prove it. If not, provide a counterexample.

Answer: Since F and G are concave functions with domain A , it must be that if $x \in A$ and $y \in A$, then for all t between 0 and 1,

$$F(tx + (1 - t)y) \geq tF(x) + (1 - t)F(y)$$

and

$$G(tx + (1 - t)y) \geq tG(x) + (1 - t)G(y)$$

Given these two inequalities, we see that

$$\begin{aligned} H(tx + (1 - t)y) &= F(tx + (1 - t)y) + G(tx + (1 - t)y) \\ &\geq tF(x) + (1 - t)F(y) + tG(x) + (1 - t)G(y) \\ &= t(F(x) + G(x)) + (1 - t)(F(y) + G(y)) \\ &= tH(x) + (1 - t)H(y) \end{aligned}$$

By definition, the left side of Expression 4 equals $H(tx + (1 - t)y)$. The right side of Expression 4 equals $tF(x) + tG(x) + (1 - t)F(y) + (1 - t)G(y) = tH(x) + (1 - t)H(y)$. Therefore

$$H(tx + (1 - t)y) \geq tH(x) + (1 - t)H(y)$$

for all t between 0 and 1, which means that H is a concave function.

3 Let F and G be real-valued concave functions with the same domain, A . Define the function H so that for all $x \in A$, $H(x) = F(x)G(x)$.

A) Is H a concave function? If so, prove it. If not, provide a counterexample.

Answer: H is not necessarily concave. For example, let A be the real line and let $F(x) = x$ and $G(x) = x$. Both F and G are concave functions. But $H(x) = x^2$. If a function of a single real variable is concave, its second derivative is negative or zero everywhere. But $H''(x) = 2 > 0$, so H is not a concave function.

B) Is H a quasi-concave function? If so, prove it. If not, provide a counterexample.

Answer: H is not in general quasi-concave, but it will be if $F(x) > 0$ and $G(x) > 0$ for all $x \in A$. If this is the case, then $\ln H(x) = \ln F(x) + \ln G(x)$. Now $\ln F(x)$ and $\ln G(x)$ are concave

functions (because a concave function of a concave function is concave). Therefore $\ln H(x)$ which is the sum of two concave functions is concave. It follows that $H(x)$ is a monotone transformation of a concave function and hence is quasi-concave.

On the other hand, if one of these functions is negative valued, $\ln H$ is not well-defined, so this argument doesn't work and in fact $H(x)$ doesn't have to be quasi-concave when F and G are concave. For example, suppose that A is the set of all real numbers and $F(x) = -1$ for all x , while and $G(x) = x^{-2}$. Both of these functions are concave, since their second derivatives are non-positive for all x . Then $H(x) = -G(x) = x^2$. This function is not quasi-concave. To see this, note that $H(1) = H(-1) = 1$, but $H\left(\frac{1}{2}1 + \frac{1}{2}(-1)\right) = H(0) = 0 < H(1)$.

4) A consumer has preferences represented by the utility function

$$U(x_1, x_2) = x_1 + x_2 + 2x_2^{1/2}.$$

Good 1 is the numeraire and has price 1. The price of good 2 is p_2 and the consumer's income is m .

A) Find this consumer's Marshallian demands for goods 1 and 2 as a function of p_2 and m . Be careful to account for corner solutions if there are any.

Answer:The marginal rate of substitution between good 2 and good 1 is $1 + x_2^{-1/2} > 1$. At an interior solution, it must be that $p_2 = 1 + x_2^{-1/2}$. This is possible only if $p_2 > 1$. If the consumer chooses positive amounts of both goods, the Marshallian demands are

$$x_2(1, p_2, m) = \left(\frac{1}{p_2 - 1}\right)^2.$$

and

$$x_1(1, p_2, m) = m - p_2 \left(\frac{1}{p_2 - 1}\right)^2.$$

There is positive consumption of good 1 if and only if $p_2 > 1$ and $m > p_2 \left(\frac{1}{p_2 - 1}\right)^2$. If consumption of good 1 is zero, then $x_2(1, p_2, m) = m/p_2$ and $x_1(1, p_2, m) = 0$. Consumption of good 2 is always positive, since the marginal rate of substitution approaches infinity as x_2 approaches zero.

B) Use your solution to Part A and the relevant homogeneity property of Marshallian demand to find this consumer's demands for goods 1 and 2 for arbitrary

non-negative prices p_1 , p_2 , and income m . (Simplify your expressions for answers as much as possible.)

Answer: Since demand is homogeneous of degree zero in prices and income, it must be that

$$x_i(p_1, p_2, m) = x_i\left(1, \frac{p_2}{p_1}, \frac{m}{p_1}\right)$$

Therefore it follows from the answers to Part A that at an interior solution,

$$\begin{aligned} x_2(p_1, p_2, m) &= \left(\frac{1}{\frac{p_2}{p_1} - 1}\right)^2 = \left(\frac{p_1}{p_2 - p_1}\right)^2 \\ x_1(p_1, p_2, m) &= \frac{m}{p_1} - \frac{p_2}{p_1} \left(\frac{1}{\frac{p_2}{p_1} - 1}\right)^2 = \frac{m}{p_1} - \frac{p_1 p_2}{p_2 - p_1} \end{aligned}$$

C) Find this consumer's Hicksian demand functions $h_1(p_1, p_2, u)$ and $h_2(p_1, p_2, u)$. Be careful to account for corner solutions if there are any.

Answer: At an interior solution,

$$h_2(p_1, p_2, u) = \left(\frac{p_1}{p_2 - p_1}\right)^2$$

and

$$\begin{aligned} h_1(p_1, p_2, u) &= u - h_2(p_1, p_2, u) - 2h_2(p_1, p_2, m)^{1/2} \\ &= u - \left(\frac{p_1}{p_2 - p_1}\right)^2 - 2\left(\frac{p_1}{p_2 - p_1}\right) \\ &= u - \left(\left(\frac{p_1}{p_2 - p_1} + 1\right)^2 - 1\right) \\ &= u + 1 - \left(\frac{p_2}{p_2 - p_1}\right)^2 \end{aligned}$$

From 4 we see that there will be an interior solution if and only if

$$u + 1 > \left(\frac{p_2}{p_2 - p_1}\right)^2.$$

If

$$u + 1 < \left(\frac{p_2}{p_2 - p_1} \right)^2,$$

then $h_1(p_1, p_2, u) = 0$ and $u = h_2(p_1, p_2, u) + 2\sqrt{h_2(p_1, p_2, u)}$. The right side of the second equation is a strictly increasing function of h_2 , which means that for any specified (p_1, p_2, u) , there is a unique solution for $h_2(p_1, p_2, u)$. However, this solution does not have a simple closed form expression.

D) Find this consumer's expenditure function $e(p_1, p_2, u)$.

Answer:

$$e(p_1, p_2, u) = p_1 h(p_1, p_2, u) + p_2 h_2(p_1, p_2, u).$$

We can write this out in detailed form using the results of Section C. For an interior solution, we have

$$\begin{aligned} e(p_1, p_2, u) &= p_1 \left(u - \left(\frac{p_2}{p_2 - p_1} \right)^2 + 1 \right) + p_2 \left(\frac{p_1}{p_2 - p_1} \right)^2 \\ &= p_1(u + 1) + \frac{p_2 p_1^2 - p_1 p_2^2}{(p_2 - p_1)^2} \\ &= p_1(u + 1) - \frac{p_1 p_2}{p_2 - p_1} \end{aligned} \tag{4}$$

E) Verify that Shephard's lemma applies in this case.

Answer: We will show this for the case of interior solutions. According to Shephard's lemma,

$$\frac{\partial e(p_1, p_2, u)}{\partial p_i} = h_i(p_1, p_2, u).$$

Differentiating the expression in Part D with respect to p_2 , we have

$$\frac{\partial e(p_1, p_2, u)}{\partial p_2} = \frac{-(p_2 - p_1)p_1 + p_1 p_2}{(p_2 - p_1)^2} = \frac{p_1^2}{(p_2 - p_1)^2}.$$

We found in part C that

$$h_2(p_1, p_2) = \frac{p_1^2}{(p_2 - p_1)^2}.$$

Thus we see that Shephard's lemma applies for good 2.

Differentiating the expression in Part D with respect to p_1 , we have

$$\begin{aligned}\frac{\partial e(p_1, p_2, u)}{\partial p_1} &= u_1 + 1 - \frac{(p_2 - p_1)p_2 + p_1p - 2}{(p_2 - p_1)^2} \\ &= u + 1 - \left(\frac{p_2}{(p_2 - p_1)}\right)^2\end{aligned}$$

We found in Part C that

$$h_1(p_1, p_2, u) = u + 1 - \left(\frac{p_2}{(p_2 - p_1)}\right)^2.$$

This verifies that Shephard's lemma holds for good 2.

5) A consumer has utility function

$$u(x_1, x_2) = \min\{v_1(x_1, x_2), v_2(x_1, x_2)\}$$

where v_1 and v_2 are both quasi-concave functions. Is u quasi-concave? If so, prove it. If not, provide a counterexample.

Answer: *Credit to Irving Fernandez and Serena Canaan for the most elegant answers to this one.*

Note that the assumption that v_i is concave means that for $i = 1, 2$,

$$v_i(tx + (1 - t)y) \geq \min\{v_i(x), v_i(y)\}.$$

Now

$$\begin{aligned}u(tx + (1 - t)y) &= \min\{v_1(tx + (1 - t)y), v_2(tx + (1 - t)y)\} \\ &\geq \min\{\min\{v_1(x), v_1(y)\}, \min\{\min\{v_2(x), v_2(y)\}\} \\ &= \min\{\min\{v_1(x), v_2(x)\}, \min\{\min\{v_1(y), v_2(y)\}\} \\ &= \min\{u(x), u(y)\}\end{aligned}$$

This implies that u is quasi-concave.

The clever key step in this argument is the step from line 2 to line 3. Can you explain why one is justified in taking this step?

6) A sculpture is placed on top of a horizontal grid. The height of the sculpture above the point on the grid with coordinates (x_1, x_2) is $x_1^2 - 3x_1x_2 + x_2^2$. An ant is crawling on the surface of the sculpture.

A) If the ant is initially at the point directly above the point $(2, 1)$ on the grid, what is the directional derivative of the height of the ant if it is crawling on the surface of the sculpture in the direction $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ as measured on the grid?

Answer: The gradient of the height of the sculpture at the point $(2, 1)$ is $(2x_1 - 3x_2, -3x_1 + 2x_2) = (1, -4)$. The directional derivative in the direction $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ is the inner product of this directional vector with the gradient. This is equal to $\frac{-7}{\sqrt{5}}$.

B) In what direction on the surface should the ant crawl if it wants to climb most steeply? (Hint: Directions should be described by a vector whose length is 1 unit.)

Answer: If the ant wants to climb most steeply, it should climb in the direction of the gradient. That direction is $(\frac{1}{\sqrt{17}}, \frac{-4}{\sqrt{17}})$.

C) In what direction should it crawl if it wants to descend most steeply?

Answer: To descend most steeply, it should crawl in the opposite direction from the gradient. This is $(\frac{-1}{\sqrt{17}}, \frac{4}{\sqrt{17}})$.

D) If the ant is moving along the surface of the statue in the direction of steepest climb at the rate of one unit per second, at what rate is its height above the ground increasing?

Answer: The answer is the inner product of the gradient with the vector representing the direction of steepest climb. The initial rate of ascent if the ant crawls in the direction of the gradient is the inner product of the gradient vector $(1, -4)$ with the direction of steepest ascent which is $(\frac{1}{\sqrt{17}}, \frac{-4}{\sqrt{17}})$. This inner product is equal to

$$1 \times \frac{1}{\sqrt{17}} + (-4) \times \frac{-4}{\sqrt{17}} = \sqrt{17}.$$