Expenditure Functions and Duality

- For a consumer with utility function $u(x)$, the Hicksian demand correspondence $h(p, u)$ maps prices and utility to the set of cheapest bundles at prices p that yield utility $h(p, u)$.
- For price vector, p and utility u, the expenditure function, $e(p, u)$, reports the lowest cost at which you could afford to achieve utility u. Thus for any $x \in h(p, u)$, $e(p, u) = px$.
- In production theory, let the production function f correspond to u. Then the expenditure function corresponds to the cost function, which reports for any vector of factor prices, and any quantity, the cost of producing that quantity in the cheapest possible way.

Properties of Expenditure Functions

- Non-decreasing in p.
- Homogeneous of degree 1 in p.
- Concave in p.
- Continuous in p for $p >> 0$.
- Shephard's Lemma

Dual Problems

- Let $u(x)$ be continuous and locally non-satiated on $\Re^n_+$.
  - (1) $px \leq m$ and $u(x) \geq u(x')$ if $px' \leq m$.
  - In words: The bundle x maximizes utility subject to $px' \leq m$. Anything better than x costs more than x.
  - (2) $u(x) = u$ and $px \leq px'$ if $u(x') \geq u(x)$.
  - In words: The bundle x minimizes cost subject to $u(x') \geq u(x)$. Anything at least as good as x costs at least as much.
- Claim: If x satisfies (1), it also satisfies (2) and if x satisfies (2), it also satisfies (1).
- Let's prove this.

Proof that if x satisfies (1) it satisfies (2)

Suppose that x does not satisfy (2). Then anything better than x costs more than m, but there is something (call it $\hat{x}$ that is at least as good as x and cheaper than m. So it would have to be that $p\hat{x} < m$ and $u(\hat{x}) = u(x)$. Since $p\hat{x} < m$, there is some ϵ neighborhood of $\hat{x}$ such that for all x' in that neighborhood $px' < m$. But we have assumed that preferences are locally non-satiated, so there must be an x^* in that neighborhood for which $u(x^*) > u(x)$ and $px^* \leq m$. This implies that x does not satisfy (1). It follows that if x satisfies (1), it must satisfy (2).

Proof that if x satisfies (2) it satisfies (1)

Suppose that $px = m > 0$ and x does not satisfy (1). Then there is some bundle $\hat{x}$ such that $u(\hat{x}) > u(x)$ and $p\hat{x} \leq m$. Since u is continuous, it will be true that be for $t < 1$ but close enough to 1 $u(x^*) > u(x)$ where $x^* = tx$. But $px^* = tpx = tm < m$. So this means that x does not satisfy Condition (1), since we have found something better than x that is cheaper than x. So if x does not satisfy (1), it does not satisfy (2). Hence if it satisfies (2) it must satisfy (1).

Some identities (when demand correspondences are single-valued)

- For any price vector p and income m, the cost of achieving utility $v(p, m)$ is m: $\quad e(p, v(p, m)) = m$
- For any price vector p and achievable utility level u, the indirect utility of price vector p and income $e(p, u)$ is u: $v(p, e(p, u)) = u$
- The vector of Marshallian demands with price vector p and income m is the same as the vector of Hicksian demands at price vector p and utility $v(p, m)$: $\quad x(p, m) = h(p, v(p, m)$
- The vector of Hicksian demands with price vector p and utility u is the same as the vector of Marshallian demands with price vector p and income $e(p, u)$: $\quad h(p, u) = x(p, e(p, u))$