Notes on General Equilibrium in an Exchange Economy

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From Demand Theory to Equilibrium Theory

We have studied Marshallian demand functions for rational consumers, where $D^i(p, m_i)$ is the vector of commodities demanded by consumer i when the price vector is p.

In general, the incomes of individuals depend on the prices of goods and services that they have to sell. Therefore in the study of general equilibrium theory, we need to make incomes depend on the prices of commodities. This is nicely illustrated in the example of a pure exchange economy where there is no production, but agents have initial endowments of goods which can they bring to market and trade with each other. Each consumer initially has some vector of endowments of goods. These goods are traded at competitive prices and in equilibrium the total demand for each good is equal to the supply of that good.

A Pure Exchange Economy

There are m consumers and n goods. Consumer i has a utility function $u^i(x^i)$ where x^i is the bundle of goods consumed by consumer i. In a competitive market, Consumer i has an initial endowment of goods which is given by the vector $\omega^i \geq 0$. Where p is the vector of prices for the n goods, consumer i's budget constraint is $px^i \leq p\omega^i$ which simply says that the value at prices p of what he consumes cannot exceed the value of his endowment.

Consumer i chooses the consumption vector $D^i(p)$ that solves this maximization problem. Where $x^i(p, m^i)$ is i's Marshallian demand curve, we have

$$D^i(p) = x^i(p, p\omega^i).$$

Let us denote i's demand for good j by $D^i_j(p)$, which is the jth component of the vector $D^i(p)$.

A pure exchange equilibrium occurs at a price $\bar{p}$ such that total demand for each good equals total supply. This means that

$$\sum_{i=1}^{m} D^i_j(\bar{p}) = \sum_{i=1}^{m} \omega^i_j$$

for all $j = 1, \ldots n$.

This vector equation can be thought of as n simultaneous equations, one for each good. Finding a competitive equilibrium price amounts to solving these n equations in n unknowns.

There are two important facts that simplify this task if the number of commodities is small.

Homogeneity and a *numeraire*

The first is that the functions $D^i(p)$ are all homogeneous of degree zero in prices and hence, so is $\sum_i D^i(p)$. To see this, note that if you multiply all prices by the same amount, you do not change the budget constraint (since if $px^i = p\omega^i$, then it must also be that $kpx^i = kp\omega^i$ for al $k > 0$. Therefore we can set one of our prices equal to 1 and solve for the remaining prices. Since any multiple of this price vector would also be a competitive equilibrium, we lose no generality in setting this price to 1.

Walras Law and one Equality for Free

The second fact is a little more subtle. It turns out that if demand equals supply for all $n-1$ goods other than the numeraire, then demand equals supply for the numeraire good as well. This means that to find equilibrium where there are n goods, we really only need to solve $n-1$ equations in $n-1$ unknowns. Thus if $n = 2$, we only need to solve a single equation. If $n = 3$, we still only need to solve 2 equations in 2 unknowns.

To see why this happens, we prove an equality that is known as Walras' Law.

If all consumers are locally nonsatiated, we know that $pD^i(p) = p\omega^i$ and so

$$p \sum_i D^i(p) = p \sum_i \omega^i$$

or equivalently,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_j (D_j^i(p) - \omega_j^i) = 0. \tag{1}$$

This equality is preserved if we reverse the order of summation, in which case we have

$$\sum_{j=1}^{n} p_j \sum_{i=1}^{m} (D_j^i(p) - \omega_j^i) = 0. \tag{2}$$

Let us define aggregate excess demand for good j as

$$E_j(p) = \sum_{i=1}^{m} \left(D_j^i(p) - \omega_j^i \right). \tag{3}$$

Then Equation 2 can be written as

$$\sum_{j=1}^{n} p_j E_j(p) = 0. \tag{4}$$

This is the equation commonly known as Walras' Law. Equation 4 implies that

$$\sum_{j \neq k} p_j E_j(p) = -p_k E_k(p). \tag{5}$$

Let good k be the numeraire. Suppose that at price vector $\bar{p}$, demand equals supply for all commodities $j \neq k$. Then $E_j(\bar{p}) = 0$ for all $j \neq k$. Therefore

$$\sum_{j \neq k} \bar{p}_j E_j(\bar{p}) = 0. \tag{6}$$

It follows from Equation 5 that

$$p_k E_k(\bar{p}) = 0.$$

But $p_k = 1$. Therefore $E_k(\bar{p}) = 0$.

Example 1

There are m consumers and two goods. Consumer i has utility function

$$U_i(x_1, x_2) = x_1^{\alpha^i} x_2^{1-\alpha^i}$$

and endowment (ω_1^i, ω_2^i). Let good 1 be the *numeraire* with price 1 and let p be the price of good 2. Then the demand function of Consumer i for good 2 is

$$D_2^i(1,p) = \frac{(1-\alpha^i)}{p}\left(\omega_1^i + p\omega_2^i\right) = \frac{1}{p}(1-\alpha^i)\omega_1^i + (1-\alpha^i)\omega_2^i \tag{7}$$

Aggregate excess demand for Good 2 is given by

$$E_2(p) = \sum_{i=1}^{n}\left(D_2^i(1,p) - \omega_2^i\right) \tag{8}$$

At a competitive equilibrium price $\bar{p}$ for good 2, it must be that $E_2(\bar{p}) = 0$. From equations 7 and 8 it follows that at a competitive equilibrium price $\bar{p}$, we have

$$\frac{1}{\bar{p}}\sum_{i=1}^{m}(1-\alpha^i)\omega_1^i + \sum_{i=1}^{m}(1-\alpha^i)\omega_2^i = \sum_{i=1}^{m}\omega_2^i \tag{9}$$

By rearranging the terms of Equation 9, we can solve for the equilibrium price $\bar{p}$ which is

$$\bar{p} = \frac{\sum_{i=1}^{m}(1-\alpha^i)\omega_1^i}{\sum_{i=1}^{m}\alpha^i\omega_2^i} \tag{10}$$

Using Walras Law, we know that when excess demand is zero for good 1, it is also zero for good 2. Therefore where $\bar{p}$ is given by Equation 10, at price vector $(\bar{p}, 1)$ we have demand equal to supply both for good 1 and for good 2.

In the special case where preferences are identical, so that $\alpha^i = \alpha$ for all i we see that the solution in Equation 10 simplifies to

$$\bar{p} = \frac{(1-\alpha)}{\alpha}\frac{\sum_i \omega_1^i}{\sum_i \omega_2^i}. \tag{11}$$

In this case, the price of good 2 is inversely proportional to the ratio of the supply of good 2 to the supply of good 1 and is directly proportional to the ratio of the Cobb-Douglas exponent on good 2 relative to that on good 1.

Notice that this would also be the solution if there were only one consumer who had an initial endowment of $\sum_i \omega_1^i$ of good 1 and $\sum_i \omega_2^i$ of good 2. In this case, at the price vector $(1, \bar{p})$ this rich consumer would demand exactly the entire amount of each good that is available.

Problems

Problem 1

There are m consumers, all of whom have identical homothetic utility functions. Note that aggregate demand is the same as it would be if one consumer had all the utility. So how can you find equilibrium prices? At what prices would this consumer demand exactly the quantities that are available in the endowment?

Special case. Suppose

$$u_i(x_1, \ldots, x_n) = \frac{1}{\alpha}\left(\sum_{j=1}^{n} a_j x_j^\alpha\right)$$

where $\alpha \leq 1$. Suppose that initial endowment of consumer i is given by the vector $\omega^i = (\omega_1^i, \ldots, \omega_n^i)$. Find an explicit solution for a competitive equilibrium price vector. Now find the quantities of each good purchased by each consumer. (Hint: Remember that their utility functions are identical and homothetic.)

Problem 2

There are three commodities in a pure exchange economy. Let good 3 be the numeraire. There are m consumers. The total endowments of goods 1, 2, and 3, are given respectively by ω_1, ω_2, and ω_3.

The aggregate demand functions for goods 1 and 2 (when ω_3 is large enough) are given as follows:

$$\begin{aligned} D^1(p_1, p_2, 1) &= a_1 - b_1 p_1 + c p_2 \\ D^2(p_1, p_2, 1) &= a_2 + c p_1 - b_2 p_2 \end{aligned} \tag{12}$$

where $a_1 > \omega_1$, $a_2 > \omega_2$ and $b_1 b_2 > c^2$.

A) Compute competitive equilibrium prices. What do we mean by ω_3 is "large enough"?

B) Can you find individual utility functions for consumers such that aggregate demand takes this form?

Hint: What if

$$u_i(x_1^i, x_2^i, x_3^i) = x_3 + a_1^i x_1^i + a_2^i x_2^i - \frac{b_1^i}{2}(x_1^i)^2 - \frac{b_2^i}{2}(x_2^i)^2 + c x_2^i x_2^i \tag{13}$$

Partial and General Equilibrium Comparative Statics

Suppose that there are n commodities in an exchange economy. For convenience, let commodity n be the numeraire. Let $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)$ be a competitive equilibrium price vector. Suppose that the demand for Good 1 is given by the function $D_1(\bar{p}_1, \ldots \bar{p}_n, \alpha)$, where α is a parameter that "shifts" the demand curve. If the aggregate endowment of good 1 is ω_1, then in competitive equilibrium it must be that

$$D_1(\bar{p}_1, \ldots \bar{p}_n, \alpha) = \omega_1 \tag{14}$$

We are interested in predicting the effects of a shift in the demand curve or of a change in the aggregate supply. Let us first consider the "partial equilibrium" approach. If we knew the demand function, what would be our prediction about the change in the equilibrium price of good 1 that would result from a change in the supply of good 1? The partial equilibrium approach is to assume that prices of all goods other than good 1 are held constant, and to see what change in p_1 is needed to reestablish equilibrium if we change the supply, ω_1. To find this, we differentiate both sides of equation 14 with respect to ω_1. When we do so, we find that

$$\frac{\partial D_1(\bar{p}_1, \ldots \bar{p}_n, \alpha)}{\partial p_1} \frac{dp_1}{d\omega_1} = 1 \tag{15}$$

From Equation 15 we find our prediction of the effect of a change in endowment on the price of good 1.

$$\frac{dp_1}{d\omega_1} = \frac{1}{\frac{\partial D_1(\bar{p}_1, \ldots \bar{p}_n, \alpha)}{\partial p_1}} \tag{16}$$

This should be no surprise to anyone who has studied elementary economics. Suppose that you draw an "inverse demand curve for good 1" curve with quantity of good 1 on the horizontal axis, and price of good 1 on the vertical axis, where the quantity corresponding to price p_1 is $D_1(p_1, \bar{p}_2 \ldots \bar{p}_n, \alpha)$. The slope of this curve is

$$\frac{1}{\frac{\partial D_1(\bar{p}_1, \ldots \bar{p}_n, \alpha)}{\partial p_1}}.$$

A vertical supply line drawn at ω_1 will intersect this curve at $D_1(\bar{p}_1, \ldots \bar{p}_n, \alpha)$. If you move the supply curve by Δ, the change in the price will be

$$\frac{\Delta}{\frac{\partial D_1(\bar{p}_1, \ldots \bar{p}_n, \alpha)}{\partial p_1}}.$$

Now let us consider the general equilibrium solution for the effect of a change in the supply of good 1. In general, a change in the price of good 1 will change demand in some of the other markets. So we need to find changes in all n prices such that demand in market 1 is changed by the amount of supply change and such that demand in all the other markets (where supply has not changed) is the same as it was before the price change.

Let us define $p(\omega)$ to be the equilibrium price vector if the vector of aggregate supplies is ω. We recall that in equilibrium it must be true that for all commodities $i = 1, \ldots, n-1$, $D_i\left(p(\omega)\right) = \omega_i$. Let us differentiate both sides of each of these $n-1$ equations with respect to ω_1. We will find that

$$\sum_{j=1}^{n-1} \frac{\partial D_1(p_1\left(\omega\right), \ldots, p_{n-1}(\omega), 1)}{\partial p_j} \left(\frac{\partial p_j(\omega)}{\partial \omega_1}\right) = 1 \tag{17}$$

and for $i = 2, \ldots, n-1$,

$$\sum_{j=1}^{n-1} \frac{\partial D_i(p_1\left(\omega\right), \ldots, p_{n-1}(\omega), 1)}{\partial p_j} \left(\frac{\partial p_j(\omega)}{\partial \omega_1}\right) = 0 \tag{18}$$

We can write Equations 17 and 18 as a matrix equation of the form

$$Mx = y$$

where M is the $n-1$ by $n-1$ matrix whose ijth entry is

$$\frac{\partial D_i(p_1\left(\omega\right), \ldots, p_{n-1}(\omega), 1)}{\partial p_j},$$

x is the column vector whose elements are $(\frac{\partial p_1}{\partial \omega_1} \dots \frac{\partial p_n}{\partial \omega_1})$, and y is the column vector whose first element is 1 and with all other elements equal to 0.

We want to solve for x which is the vector of changes in each of the prices. If M has an inverse, this is $x = M^{-1}y$. But since y is a vector whose first element is 1 and all other elements are zero, we see that the vector x is just the first column of the matrix M^{-1}.

Example

Consider a three good example like that we considered in Problem 2. The demand functions for goods 1 and 2

$$
\begin{aligned}
D^1(p_1, p_2, 1) &= a_1 - b_1 p_1 + c p_2 \\
D^2(p_1, p_2, 1) &= a_2 + c p_1 - b_2 p_2
\end{aligned}
\tag{19}
$$

If $c > 0$, the goods are substitutes. If $c < 0$, the goods are complements. The matrix of partial derivatives is

$$M = \begin{pmatrix} -b_1 & c \\ c & -b_2 \end{pmatrix}$$

and so

$$M^{-1} = \frac{1}{b_1 b_2 - c^2} \begin{pmatrix} -b_2 & -c \\ -c & -b_1 \end{pmatrix}$$

Therefore

$$\frac{\partial p_1(\omega_1, \omega_2)}{\partial \omega_1} = \frac{-b_2}{b_1 b_2 - c^2}$$

and

$$\frac{\partial p_2(\omega_1, \omega_2)}{\partial \omega_1} = \frac{-c}{b_1 b_2 - c^2}.$$

In the special case where $c = 0$, these imply that

$$\frac{\partial p_1(\omega_1, \omega_2)}{\partial \omega_1} = \frac{-1}{b_1}$$

and

$$\frac{\partial p_2(\omega_1, \omega_2)}{\partial \omega_1} = 0.$$

These are the partial equilibrium answers, which will differ from the general equilibrium answers if there are cross effects.

We see that whether the goods are substitutes or complements, the existence of cross-effects amplifies the effect of supply changes. If the goods are substitutes, an increase in the supply of good 1 will drive the price of good 1 down, but a fall in the price of good 1 reduces demand for good 2 and this will require a reduction in the price of good 2 to clear market 2. But this price reduction in market 2 will reduce demand in market 1 and so the price in market 2 will have to be reduced further to equilibrate supply and demand in market 1. But this means that the price in c market 2 has to be reduced further. And so it goes...But if $b_1 b_2 - c^2 > 0$, this process settles down.

If the goods are complements, an increase in the supply of good 1 drives the price of good 1 down, but this increases the demand for its complement, good 2. The increase in demand for good 2 means that the price of good 2 has to increase to maintain equilibrium. in market 2. This reduces demand for good 1 and so a further price reduction is needed to achieve equilibrium in market 1. And so it goes...

Problem 3

A pure exchange economy has N_1 Type 1 consumers and N_2 type 2 consumers. There are two goods, X and Y. Consumers of Type 1 each have an initial endowment of ω_y units of good Y and none of good X. Type 2's each have an initial endowment of ω_x units of good X and none of good Y. Each consumer of type i has preferences over consumption bundles (x, y) that are represented by the utility function

$$u_i(x, y) = x^{\alpha_i} y^{(1-\alpha_i)}.$$

Let Good X be the numeraire.

A) Solve for the quantity of each good demanded by consumers of each type when the price of good Y is p. Let Good X be the numeraire.

B) Solve for a competive equilibrium price as a function of the prarmeters N_1, N_2, α_1, α_2, ω_1 and ω_2.

C) Find an expression for the utility of each type of consumer in competitive equilibrium as a function of these parameters. Comment on the qualitative nature of the effect of these parameters on individual welfare. Who gains and who loses from changes in the various parameters?

Income effects and multiple equilibria

This example is inspired by a paper by Lloyd Shapley and Martin Shubik from the *Journal of Political Economy* in 1977. A more thorough treatment appears in "Simple economies with multiple equilibria" by Ted Bergstrom, Ken-Ichi Shimomura and Takehiko Yamato, found at `http://escholarship.org/uc/item/6qv909xs#page-2`

A Shapley-Shubik economy is a pure exchange economy with two consumers and two goods X and Y. Both consumers have quasi-linear preferences, but their preferences are linear in different goods. Each consumer's initial endowment includes positive amounts only of the good in which he has linear utility. Thus the utility functions of consumers 1 and 2 are:

$$\begin{aligned} U_1(x_1, y_1) &= x_1 + f_1(y_1) \\ U_2(x_2, y_2) &= y_2 + f_2(x_2) \end{aligned} \tag{20}$$

where $f_1(\cdot)$ and $f_2(\cdot)$ are increasing, concave functions. The initial endowment of consumer 1 is $(x^0, 0)$ and the initial endowment of consumer 2 is $(0, y^0)$.

You may ask, but if there are just two consumers and each one is a monopolist of one of the goods, why should one think they would trade at competitive prices? Probably they would not. But suppose that the economy consisted of 1000 people just like consumer 1 and 1000 people just like consumer 2. The competitive equilibrium prices and the equilibrium consumptions that we would find for each of our two consumers would be the same for each of the 2000 consumers in the big economy. In this economy each consumer has 999 "competitors" offering the same products in the market.

Let us consider a special case of a Shapley-Shubik economy where the functions f_1 and f_2 are quadratic and are "mirror-images" of each other. In particular, suppose that for some $a > 0$,

$$U_1(x, y) = x + ay - \frac{1}{2}y^2$$

for all $y \leq a$.

$$U_2(x, y) = y + ax - \frac{1}{2}x^2$$

for all $x \leq a$.

We can make good X the numeraire by setting its price to one and solving for the excess demand functions as a function of the price p of good Y. If consumer 1 demands positive amounts of both goods at price p, it must be that consumer 1's marginal rate of substitution between good Y and good X is equal to p. This gives us the equation $a - y_1 = p$, where y_1 is the amount of Y consumed by consumer 1. It follows that consumer 1's demand for good Y at price p is

$$D_Y^1(p) = a - p \tag{21}$$

If consumer 2 demands positive amounts of both goods, then it must be that consumer 2's marginal rate of substitution between good Y and good X is equal to p. This requires that

$$\frac{1}{a - x_2} = p$$

where x_2 is the amount of good 2 demanded by person 2. Therefore consumer 2's demand for X is given by

$$D_X^2 = a - \frac{1}{p}. \tag{22}$$

To find consumer 2's demand for Y, we make use of consumer 2's budget constraint. Recall that consumer 2's initial holdings vector is $(0, y_0)$, so 2's budget constraint is

$$x_2 + py_2 = py^0. \tag{23}$$

From Equations 22 and 23 it follows that

$$D_Y^2(p) = y^0 - \frac{1}{p}D_X^2(p) = y^0 - \frac{a}{p} + \frac{1}{p^2}. \tag{24}$$

If p is a competitive equilibrium price, excess demand for Y must be 0. Since Consumer 2 is the only one who has a positive initial endowment of good 2, the total supply of good 2 is y^0. Therefore

$$0 = E_Y(p) = D_Y^1(p) + D_Y^2(p) - y^0 \tag{25}$$

Substituting from equations 21 and 24 into equation 25 we find that at a competitive equilibrium,

$$E_Y(p) = a - p + y^0 - \frac{a}{p} + \frac{1}{p^2} - y^0 = 0 \tag{26}$$

Equation 26 simplifies to

$$E_Y(p) = a - p - \frac{a}{p} + \frac{1}{p^2} = 0 \tag{27}$$

We can get some qualitative information about E_Y by seeing what happens when p is small and when p is large. Note that $\lim_{p\to 0} E_Y(p) = \infty$ and that $\lim_{p\to\infty} E_Y(p) = -\infty$. Note also that $E_Y(\cdot)$ is a continuous function. Draw a diagram and you will see that there must be at least one solution. The symmetry of the problem suggests one possible solution. It is easily verified that $E(1) = 0$. Are there other solutions? (One trick that you could use is to note that if $E'(1) > 0$, there must be more solutions besides $p = 1$. Look at a graph to see why. For what values of a is $E'(1) > 0$?)[1]

For this function, there is a good direct way to find out whether there are other solutions and what they are. If we multiply both sides of equation 27 by p^2, we see that Equation 27 is satisfied only if

$$p^3 - ap^2 + ap - 1 = 0. \tag{28}$$

We see that this is a cubic equation, and as such will have at most three solutions. We already know that one of the solutions is $p = 1$, so we know that equation 28 can be factored to be expressed as $p - 1$ times a quadratic. A bit of fiddling (long division) shows that

$$p^3 - ap^2 + ap - 1 = -(p-1)\left(p^2 + (1-a)p + 1\right) \tag{29}$$

Therefore the other two roots of the equation

$$p^3 - ap^2 + ap - 1 = 0$$

are found by applying the quadratic formula to the quadratic equation

$$\left(p^2 + (1-a)p + 1\right) = 0.$$

We see that this expression will have two roots $\bar{p}$ and $\frac{1}{\bar{p}}$ if and only if $a > 3$. These are the solutions.

[1]Another trick that you could employ is to note that for $p \neq 1$, it must be that with this excess demand function, if $E(p) = 0$, then $E(1/p) = 0$ and so any solutions other than $p = 1$ come in reciprocal pairs.

Working backwards is even easier. We can choose the parameter a to give us three equilibria with equilibrium prices $\bar{p}$, 1, and $1/\bar{p}$, where $\bar{p}$ is any positive number we want. Suppose, for example, we want to have three solutions, 1/2, 1, and 2. We need to have

$$\left(p^2 + (1-a)p + 1\right) = 0$$

which is equivalent to $a = 1 + p + 1/p$. If $p = 1/2$, this is the case when

$$a = 1 + \frac{1}{2} + 2 = 3\frac{1}{2}$$