Continuous preferences in Euclidean space

- Assume that the choice set X is a subset of Euclidean n-space $\Re^n$
- For any $x \in X$, define the set $\succeq(x) = \{x' | x' \succeq x\}$ and define the set $\preceq(x) = \{x' | x \succeq x'\}$. In English, we can call these the at-least-as-good-as-x set and the no-better-than-x set.
- Preferences are said to be continuous on X, if for all $x \in X$, $\succeq(x)$ and $\preceq(x)$ are closed sets.
- What is a closed set?
    - Look at limit definition
    - Complement of an open set
        - What is an open set in $\Re^n$? ϵ-ball definition. Draw pictures.

More on continuous preferences

- Define $\succ(x) = \{x' | x' \succ x\}$ and $\prec(x) = \{x' | x \succ x'\}$. (These are the better-than-x sets and the worse-than-x sets.)
- Suppose preference are complete, then $\succeq(x)^c = \prec(x)$. (Why?)
- So if preferences are continuous and $x \succ y$, then there is a little open neighborhood of x such that everything in that neighborhood is better than y and a little neighborhood of y such that everything in that neighborhood is worse than x.

Some Examples

- Suppose that fractional goods are useless. Example: Preferences defined on $X = [0,3] \times [0,3]$ and represented by $u(x_1, x_2) = \lfloor x_1 \rfloor \lfloor x_2 \rfloor$, where $\lfloor x \rfloor$ is the greatest integer that is no bigger than x. Draw indifference map. Show that these preferences are not continuous.
- Lexicographic preferences on $\Re^2$. $(x_1, x_2) \succ (x_1', x_2')$ if $x_1 > x_1'$ or if $x_1 = x_1'$ and $x_2 > x_2'$. Draw indifference map. Show that preferences are not continuous.

A curious fact

There is no utility function that represents lexicographic preferences. There are just not enough real numbers to do so. Note that if there were, there would be an open interval of real numbers corresponding to each real number. That would mean that there is a rational number corresponding to each real. But there isn't. (A standard mathematical result.)

Continuous preferences and continuous utility (Debreu's theorem)

- If preferences $\succeq$ are complete, transitive, and continuous on X, where $X \subset \Re^n$, then there exists a continuous utility function defined on X, that represents $\succeq$.

Non-empty choice function and utility

- The choice set generated by $\succeq$ is defined so that $c(A) = \{x \in A | x \succeq x' \text{for all } x' \in A\}$
- If $\succeq$ is represented by a utility function on X, then $\succeq$ is transitive and complete on X.
- Therefore if $B \subset X$ is a finite set, and $\succeq$ is represented by a utility function, then $c(B)$ is not empty.
- But what if B is an infinite set? When is $c(B)$ not empty?

Some Examples

- Suppose preferences are defined on $\Re_2^+$, the non-negative orthant of the Euclidean plane and are represented by the utility function $u(x_1, x_2) = x_1 x_2$. Let $B = \{(x_1, x_2) \in X | x_1 \geq x_2\}$. Is $c(B)$ non-empty?
- Suppose preferences are defined on $\Re_2^+$, the non-negative orthant of the Euclidean plane and are represented by the utility function $u(x_1, x_2) = x_1 x_2$. Let $B = \{(x_1, x_2) \in X | x_1 < 2 \text{and } x_2 \leq 2\}$. Is $c(B)$ non-empty?

Maximum theorem

- A closed bounded set in $\Re_n$ is compact. (Compactness is defined for other topologies, but we will concentrate on $\Re_n$, so you can treat this as a definition.)
- Continuous functions take maxima on compact sets. Topological theorem.
- Let B be a closed, bounded subset of $X \subset \Re_n$ and let u be a continuous function defined on X. Then for the preference relation $\succeq$ that is represented by u, the set $c(B)$ is non-empty.