When is a CES function concave?

Consider a constant-elasticity-of-substitution function with constant returns:

$$f(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} \lambda_i x_i^{\rho} \right)^{\frac{1}{\rho}}. \tag{1}$$

This function will be concave if $\lambda_i \geq 0$ for all i and $\rho \leq 1$.. We could prove this by by showing that the Hessian is negative semi-definite, but let's try another method.

Step 1-Show that f is quasi-concave Let us first show that the this function is quasi-concave. A function is a quasi-concave function if it is a monotone increasing function of a concave function. So let's look for a simple concave function "hidden inside" of f.

We note that Then

$$f(x) = g(x)^{\frac{1}{\rho}} \tag{2}$$

where

$$g(x) = \sum_{i=1}^{n} \lambda_i x_i^{\rho}. \tag{3}$$

Suppose that $0 < \rho \leq 1$. Then we see from Equation 2 that $f(x)$ is a monotone increasing function of $g(x)$. Now it is easy to verify if $0 < \rho \leq 1$, then $g(x)$ is a concave function, because its Hessian is simply a diagonal matrix with entries of the form

$$\lambda_i \rho(\rho - 1)x^{\rho-2}.$$

When $0 < \rho \leq 1$, these terms are all non-positive. Therefore $f(x)$ is a monotone increasing function of a concave function $g(x)$ and hence f is concave.

Suppose that $\rho < 0$. Then it must be that $f(x) = g(x)^{1/\rho}$ is a monotone *decreasing* function of $g(x)$. But if it is a monotone decreasing function of $g(x)$, it is a monotone increasing function of $-g(x)$. Now when $\rho < 0$, the Hessian matrix of $-g(x)$ is a diagonal matrix with entries of the form

$$-\lambda_i \rho(\rho - 1)x^{\rho-2}.$$

When $\rho < 0$, these terms are all non-positive and hence $-g$ is a concave function. Then $f(x)$ is a monotone increasing function of $-g(x)$, it must be that f is quasi-concave.

Step 2-Show that f is concave

We know that f is quasi-concave, but a quasi-concave function that is homogeneous of degree 1 must be concave. You will find a proof of this proposition in the notes on "useful properties of quasi-concave and homogeneous functions" appearing in week 5.

Step 3- Generalize to CES functions that are homogeneous of degree less than 1

Where $f(x)$ is the constant returns to scale function defined in Equation 1, the CES functions that are of degree k less than 1 take the form $f(x)^k$ where $0 < k < 1$. Now if we have a concave function $f : \Re^n \to \Re$, and an increasing concave function $g \to \Re \to \Re$, then if we define the function $h : \Re^b \to \Re$ so that $h(x) = g(f(x))$, then h must be a concave function. (This has an easy proof that you should be able to supply.)