CHAPTER 16

Quadratic Forms and Definite Matrices

The natural starting point for the study of optimization problems is the simplest such problem: the optimization of a *quadratic form*. There are a number of good reasons for studying *quadratic* optimization problems first. Quadratic forms are the simplest functions after linear ones. Like linear functions, they have matrix representations, so that studying the properties of a quadratic form reduces to studying properties of a symmetric matrix. Quadratic forms provide an excellent introduction to the vocabulary and techniques of optimization problems. Furthermore, the second order conditions that distinguish maxima from minima in economic optimization problems are stated in terms of quadratic forms. Finally, a number of economic optimization problems have a quadratic objective function, for example, risk minimization problems in finance, where riskiness is measured by the (quadratic) variance of the returns from investments.

Example 16.1 Among the functions of one variable, the simplest functions with a unique global extremum are the pure quadratics: $y = x^2$ and $y = -x^2$. The former has a global minimum at $x = 0$; the latter has a global maximum at $x = 0$, as illustrated in Figure 16.1.

16.1 QUADRATIC FORMS

Recall the definition of a quadratic form on $\mathbf{R}^n$ from Section 13.3.

Definition A **quadratic form** on $\mathbf{R}^n$ is a real-valued function of the form

$$Q(x_1, \dots, x_n) = \sum_{i \le j} a_{ij} x_i x_j, \tag{1}$$

in which each term is a monomial of degree two.

The presentation in Section 13.3 showed that each quadratic form Q can be represented by a *symmetric* matrix A so that

$$Q(\mathbf{x}) = \mathbf{x}^T \cdot A \cdot \mathbf{x}. \tag{2}$$

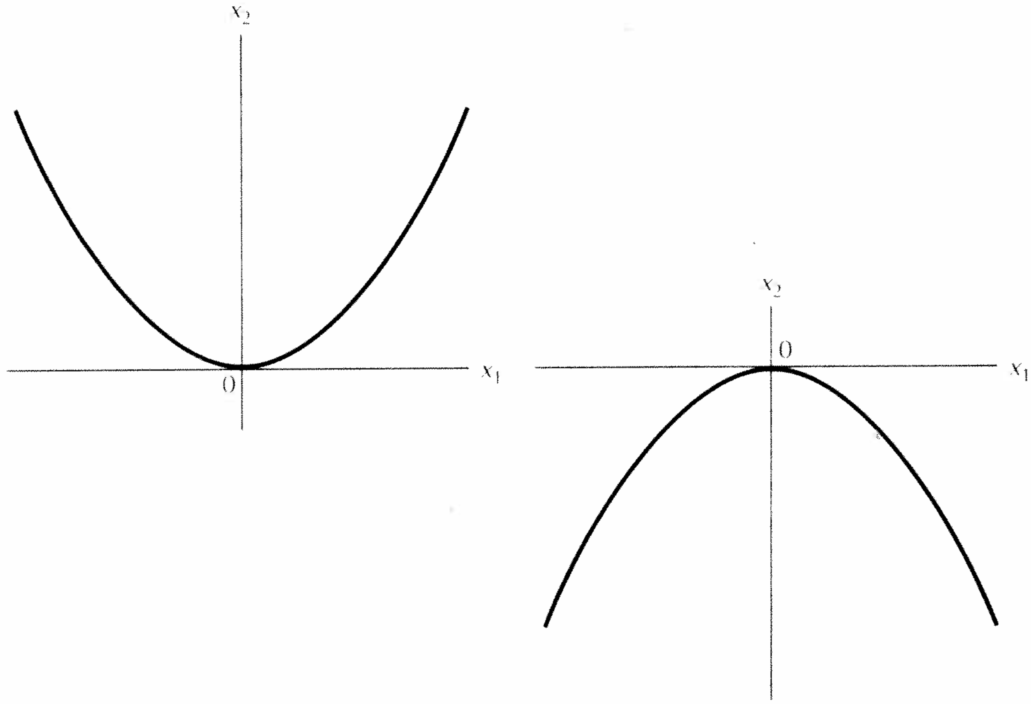


Figure 16.1 *The functions of $f(x) = x^2$ and $f(x) = -x^2$.*

For example, the general two-dimensional quadratic form

$$a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 \tag{3}$$

can be written as

$$(x_1 \quad x_2)\begin{pmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

And the general three dimensional quadratic form

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3 \tag{4}$$

can be written as:

$$(x_1 \quad x_2 \quad x_3)\begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

16.2 DEFINITENESS OF QUADRATIC FORMS

A quadratic form always takes on the value zero at the point $\mathbf{x} = \mathbf{0}$. Its distinguishing characteristic is the set of values it takes when $\mathbf{x} \neq \mathbf{0}$. In this chapter, we focus on the question of whether $\mathbf{x} = \mathbf{0}$ is a max, a min or neither of the quadratic forms under consideration.

The general quadratic form of one variable is $y = ax^2$. If $a > 0$, then ax^2 is always ≥ 0 and equals 0 only when $x = 0$. Such a form is called **positive definite**; $x = 0$ is its global *minimizer*. If $a < 0$, then ax^2 is always ≤ 0 and equals 0 only when $x = 0$. Such a quadratic form is called **negative definite**; $x = 0$ is its global *maximizer*. Figure 16.1 illustrates these two situations.

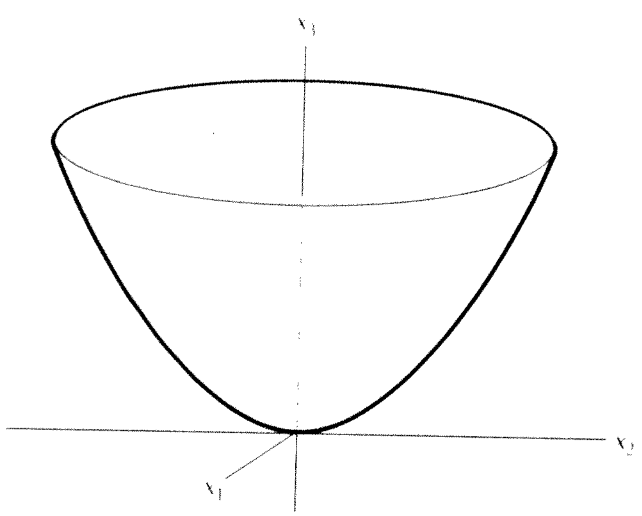
In two dimensions, the quadratic form $Q_1(x_1, x_2) = x_1^2 + x_2^2$ is always greater than zero at $(x_1, x_2) \neq (0, 0)$. So, we call Q_1 **positive definite**. Quadratic forms like $Q_2(x_1, x_2) = -x_1^2 - x_2^2$, which are strictly negative except at the origin, are called **negative definite**. Quadratic forms like $Q_3(x_1, x_2) = x_1^2 - x_2^2$, which take on both positive and negative values ($Q_3(1, 0) = +1$ and $Q_3(0, 1) = -1$) are called **indefinite**.

There are two intermediate cases: a quadratic form which is always ≥ 0 but may equal zero at some nonzero $\mathbf{x}$'s is called **positive semidefinite**. This property is illustrated by the quadratic form

$$Q_4(x_1, x_2) = (x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2,$$

which is never negative but which equals zero at nonzero points such as $(x_1, x_2) = (1, -1)$ or $(-2, 2)$. A quadratic form like $Q_5(x_1, x_2) = -(x_1 + x_2)^2$, which is never positive but can be zero at points other than the origin, is called **negative semidefinite**.

Figures 16.2 through 16.6 present the graphs of the above five quadratic forms $Q_1, \ldots, Q_5$. Every quadratic form on $\mathbf{R}^2$ has a graph similar to one of these five. For example, every positive definite quadratic on $\mathbf{R}^2$ has a bowl shaped graph as in Figure 16.2, and every indefinite quadratic on $\mathbf{R}^2$ has a saddle-shaped graph as in Figure 16.4.



Graph of the positive definite form $Q_1(x_1, x_2) = x_1^2 + x_2^2$.

Figure 16.2

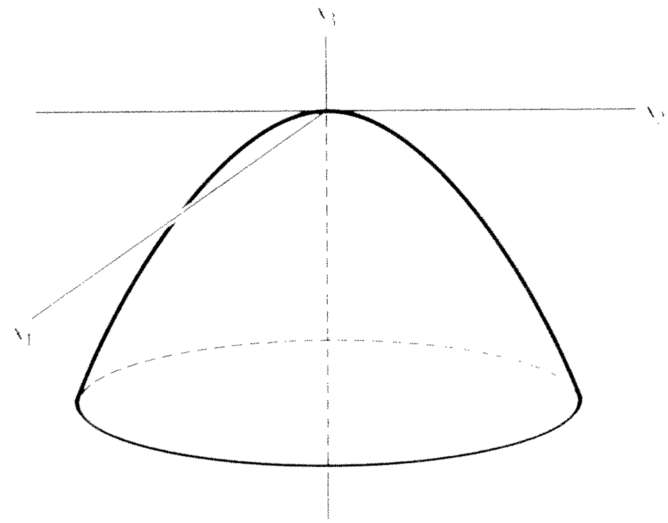


Figure 16.3 *Graph of the negative definite form* $Q_2(x_1, x_2) = -x_1^2 - x_2^2$.

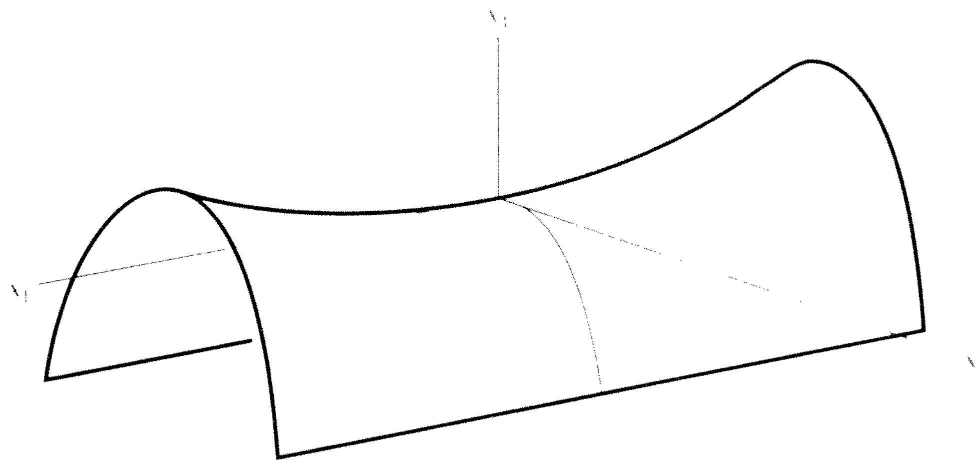


Figure 16.4 *The graph of the indefinite form* $Q_3(x_1, x_2) = x_1^2 - x_2^2$.

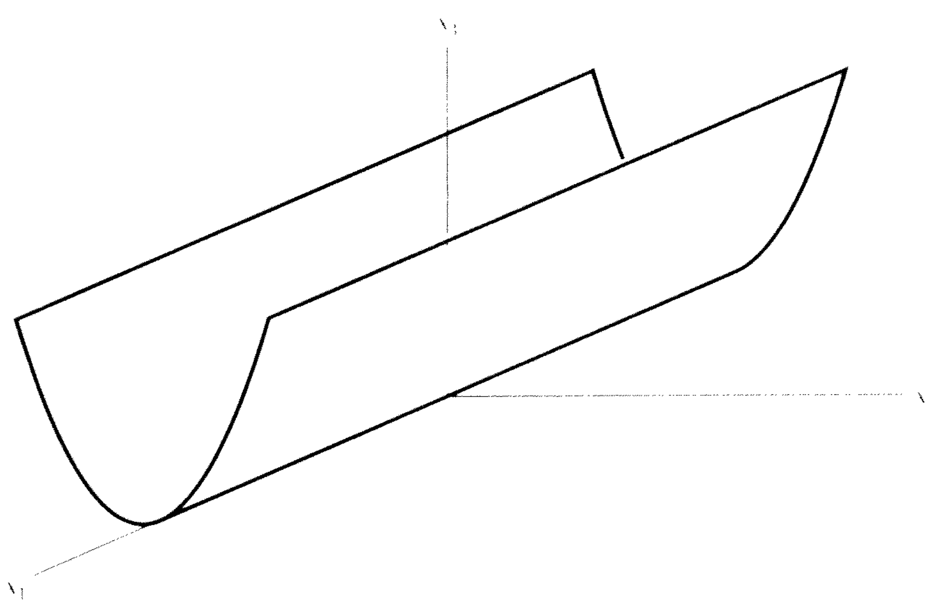
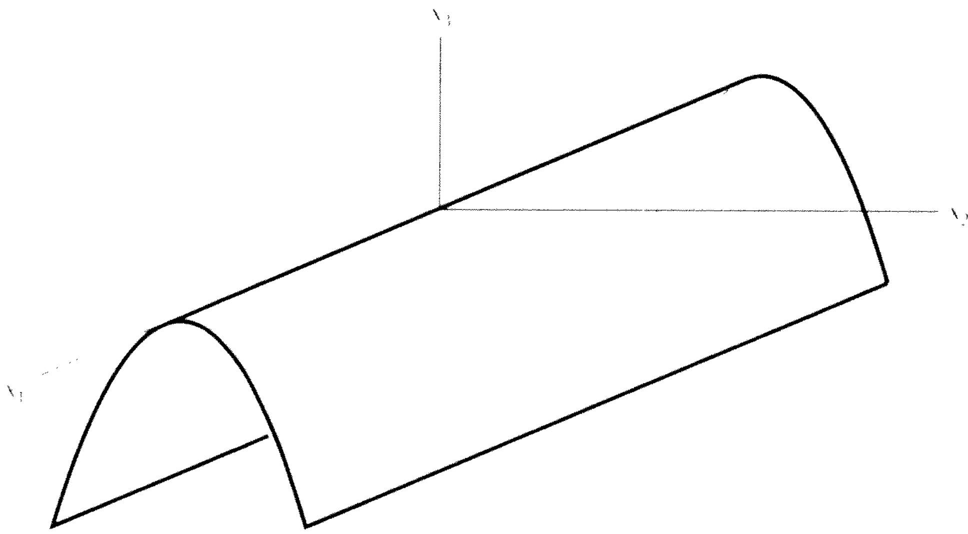


Figure 16.5 *The graph of the positive semidefinite form* $Q_4(x_1, x_2) = (x_1 + x_2)^2$.



The graph of the negative semidefinite form $Q_5(x_1, x_2) = -(x_1 + x_2)^2$.

Figure 16.6

Definite Symmetric Matrices

A symmetric matrix is called positive definite, positive semidefinite, negative definite, etc., according to the definiteness of the corresponding quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Since we will usually be applying this terminology to symmetric matrices directly, we focus on such matrices for our formal definitions of definiteness.

Definition Let A be an $n \times n$ symmetric matrix, then A is:

(*a*) **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R^n}$,
(*b*) **positive semidefinite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R^n}$,
(*c*) **negative definite** if $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R^n}$,
(*d*) **negative semidefinite** if $\mathbf{x}^T A\mathbf{x} \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R^n}$, and
(*e*) **indefinite** if $\mathbf{x}^T A\mathbf{x} > 0$ for some $\mathbf{x}$ in $\mathbf{R^n}$ and < 0 for some other $\mathbf{x}$ in $\mathbf{R^n}$.

Remark A matrix that is positive (negative) definite is automatically positive (negative) semidefinite. Otherwise, every symmetric matrix falls into one of the above five categories.

Application: Second Order Conditions and Convexity

The definiteness of a symmetric matrix plays an important role in economic theory and in applied mathematics in general. For example, for a function $y = f(x)$ of one variable, the sign of the second derivative $f''(x_0)$ at a critical point x_0 of f gives a necessary condition and a sufficient condition for determining whether x_0 is a maximum of f, a minimum of f, or neither. The generalization of this second derivative test to higher dimensions involves checking whether the

second derivative matrix (or Hessian) of f is positive definite, neg
or indefinite at a critical point of f.

In a similar vein, a function $y = f(x)$ of one variable is concav
derivative $f''(x)$ is ≤ 0 on some interval. The generalization of this r
dimensions states that a function is concave on some region if its sec
matrix is *negative semidefinite* for all $\mathbf{x}$ in the region.

Application: Conic Sections

In plane geometry, the conic section described by the level curve

$$Q(x_1, x_2) \equiv a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 = 1$$

is completely determined by the definiteness of Q or of its associat

$$A = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22} \end{pmatrix}.$$

Figure 16.7 illustrates the connection. The horizontal plane $\{x_3 = 1\}$
in Figure 16.2 in an ellipse or circle. Therefore, if A is positive defir
is an ellipse or circle. Since $\{x_3 = 1\}$ cuts the graph in Figure 16.4 in
as also illustrated in Figure 16.7, equation (5) describes an hyp

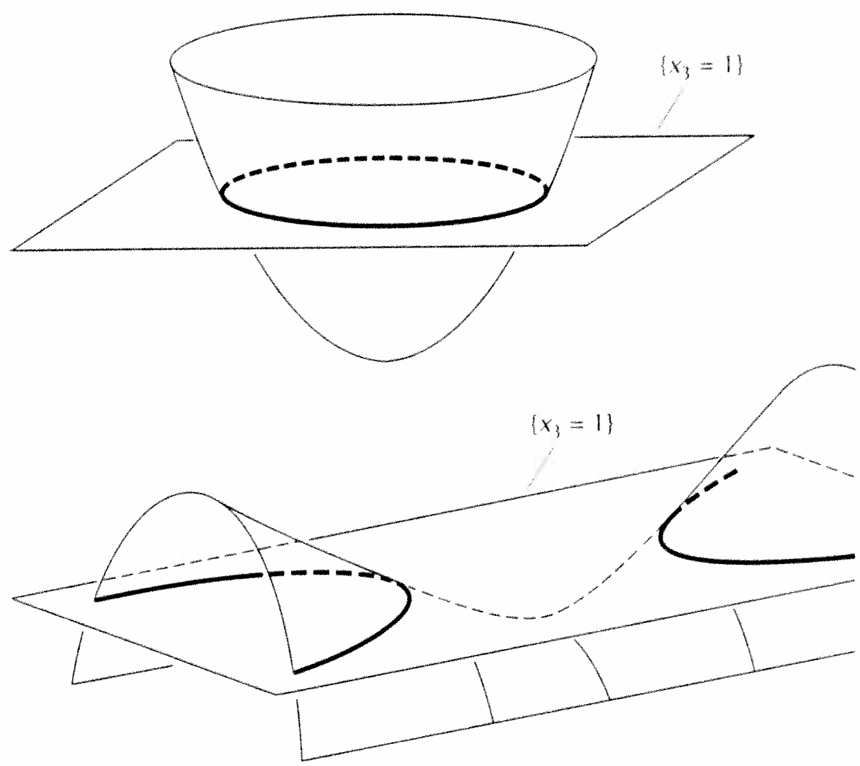


Figure 16.7 *Levels sets of graphs of quadratic forms.*

indefinite. Since $\{x_3 = 1\}$ cuts the graph of Figure 16.5 in a pair of parallel lines, equation (5) defines two lines if A is positive semidefinite but not positive definite. Finally, since the plane $\{x_3 = 1\}$ lies strictly above the graphs in Figures 16.3 and 16.6, the set (5) is empty if A is negative definite or even negative semidefinite.

Principal Minors of a Matrix

In this section we will describe a simple test for the definiteness of a quadratic form or of a symmetric matrix. To describe this algorithm, we need some more vocabulary.

Definition Let A be an $n \times n$ matrix. A $k \times k$ submatrix of A formed by deleting $n - k$ columns, say columns $i_1, i_2, \ldots, i_{n-k}$ *and the same* $n - k$ rows, rows $i_1, i_2, \ldots, i_{n-k}$, from A is called a kth order **principal submatrix** of A. The determinant of a $k \times k$ principal submatrix is called a kth order **principal minor** of A.

Example 16.2 For a general 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

there is one third order principal minor: $\det(A)$. There are three second order principal minors:

(1) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, formed by deleting column 3 and row 3 from A;

(2) $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$, formed by deleting column 2 and row 2 from A;

(3) $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, formed by deleting column 1 and row 1 from A.

There are three first order principal minors:

(1) $|\, a_{11} \,|$, formed by deleting the last 2 rows and columns,
(2) $|\, a_{22} \,|$, formed by deleting the first and third rows and the first and third columns, and
(3) $|\, a_{33} \,|$, formed by deleting the first 2 rows and columns.

It is important to understand why no other submatrix of A is a principal submatrix. For practice, list all the principal minors of a general 4×4 matrix.

Among the kth order principal minors of a given matrix, there is one special one that we want to highlight.

Definition Let A be an $n \times n$ matrix. The kth order principal submatrix of A obtained by deleting the *last* $n - k$ rows and the *last* $n - k$ columns from A is called the kth order **leading principal submatrix** of A. Its determinant is called the kth order **leading principal minor** of A. We will denote the kth order leading principal submatrix by A_k and the corresponding leading principal minor by $|A_k|$.

An $n \times n$ matrix has n leading principal submatrices — the top-leftmost 1×1 submatrix, the top-leftmost 2×2 submatrix, etc. For the general 3×3 matrix of Example 16.2, the three leading principal minors are

$$|\,a_{11}\,|, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

The following theorem provides a straightforward algorithm which uses the leading principal minors to determine the definiteness of a given matrix. We present its proof in the Appendix of this chapter. We will present other criteria for the definiteness of a symmetric matrix in Section 23.7.

Theorem 16.1 Let A be an $n \times n$ symmetric matrix. Then,

(*a*) A is positive definite if and only if all its n leading principal minors are (strictly) positive.

(*b*) A is negative definite if and only if its n leading principal minors alternate in sign as follows:

$$|A_1| < 0, \quad |A_2| > 0, \quad |A_3| < 0, \quad \text{etc.}$$

The kth order leading principal minor should have the same sign as $(-1)^k$.

(*c*) If some kth order leading principal minor of A (or some pair of them) is nonzero but does not fit either of the above two sign patterns, then A is indefinite. This case occurs when A has a *negative* kth order leading principal minor for an *even* integer k or when A has a negative kth order leading principal minor and a positive ℓth order leading principal minor for two distinct *odd* integers k and ℓ.

One way that the leading principal minor test of Theorem 16.1 can fail for a given symmetric matrix A is that some leading principal minor of A is *zero* while the nonzero ones fit the sign pattern in either *a*) or *b*) of Theorem 16.1. When this occurs, the matrix A is *not* definite and it may or it may not be semidefinite. In this case, to check for semidefiniteness, one no longer has the luxury of checking only the n *leading* principal minors of A, but must check the sign of *every* principal minor of A, using the test described by the following theorem.

Theorem 16.2 Let A be an $n \times n$ symmetric matrix. Then, A is positive semidefinite if and only if every principal minor of A is ≥ 0; A is negative semidefinite if and only if every principal minor of odd order is ≤ 0 and every principal minor of even order is ≥ 0.

Example 16.3 Suppose A is a 4×4 symmetric matrix and, as usual, write $|A_i|$ for its ith order *leading* principal minor.

(*a*) If $|A_1| > 0$, $|A_2| > 0$, $|A_3| > 0$, $|A_4| > 0$, then A is positive definite (and conversely).

(*b*) If $|A_1| < 0$, $|A_2| > 0$, $|A_3| < 0$, $|A_4| > 0$, then A is negative definite (and conversely).

(*c*) If $|A_1| > 0$, $|A_2| > 0$, $|A_3| = 0$, $|A_4| < 0$, then A is indefinite because of A_4.

(*d*) If $|A_1| < 0$, $|A_2| < 0$, $|A_3| < 0$, $|A_4| < 0$, then A is indefinite because of A_2 (and A_4).

(*e*) If $|A_1| = 0$, $|A_2| < 0$, $|A_3| > 0$, $|A_4| = 0$, then A is indefinite because of A_2.

(*f*) If $|A_1| > 0$, $|A_2| = 0$, $|A_3| > 0$, $|A_4| > 0$, then A is not definite. It is not negative semidefinite, but it may be positive semidefinite. To check for positive semidefiniteness, one must check all 15 principal minors of A, not just the four leading principal ones. If none of the principal minors are negative, then A is positive semidefinite. If at least one of them is negative, A is indefinite.

(*g*) If $|A_1| = 0$, $|A_2| > 0$, $|A_3| = 0$, $|A_4| > 0$, then A is not definite, but it may be positive semidefinite or negative semidefinite. To decide, one must again check all 15 of its principal minors.

To motivate these two theorems and to understand their algorithms better, we will examine them in some detail for the simplest symmetric matrices — diagonal matrices and then 2×2 matrices.

The Definiteness of Diagonal Matrices

The simplest $n \times n$ symmetric matrices are the diagonal matrices. They also correspond to the simplest quadratic forms since

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{6}$$

$$= a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2,$$

a sum of squares. Clearly, this quadratic form will be positive definite if and only if all the a_i's are positive and negative definite if and only if all the a_i's are negative. It will be positive semidefinite if and only if all the a_i's are ≥ 0 and negative semidefinite if and only if all the a_i's are ≤ 0. If there are two a_i's of opposite signs, this form will be indefinite.

Since all the principal submatrices are diagonal matrices, their determinants — the principal minors — are just products of the a_i's. If all the a_i's are positive, then all their products are positive and so all the leading principal minors are positive. If all the a_i's are negative (so the form is negative definite), then products of odd numbers of the a_i's will be negative and products of even numbers of the a_i's will be positive. This corresponds to the alternating signs condition in *b*) of Theorem 16.1 and indicates why we should expect such an alternating sign condition instead of an all negative condition in the test for negative definiteness.

If a_1 is zero in (6), the form cannot be definite since it will be zero when evaluated at $(1, 0, \ldots, 0)$. Notice that in this diagonal case, all the *leading* principal minors of (6) will also be zero, independent of the signs of the other a_i's. In order to check that all the a_i's have the proper sign when some of them are zero, one must check much more than just the leading principal minors.

The Definiteness of 2 × 2 Matrices

One can verify Theorems 16.1 and 16.2 directly for 2×2 symmetric matrices by completing the square in the corresponding quadratic form. Consider the general quadratic form on $\mathbf{R}^2$:

$$\begin{aligned} Q(x_1, x_2) &= (x_1 \quad x_2)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= ax_1^2 + 2bx_1x_2 + cx_2^2. \end{aligned} \tag{7}$$

For ease of notation, we are using a, b, and c in this example in place of a_{11}, a_{12}, and a_{22}, respectively. If $a = 0$, then Q cannot be positive or negative definite because $Q(1, 0) = 0$. So, assume now that $a \neq 0$ and complete the square in (7) by adding and subtracting $b^2x_2^2/a$ in expression (7):

$$\begin{aligned} Q(x_1, x_2) &= ax_1^2 + 2bx_1x_2 + cx_2^2 + \frac{b^2}{a}x_2^2 - \frac{b^2}{a}x_2^2 \\ &= a\left(x_1^2 + \frac{2b}{a}x_1x_2 + \frac{b^2}{a^2}x_2^2\right) - \frac{b^2}{a}x_2^2 + cx_2^2 \\ &= a\left(x_1 + \frac{b}{a}x_2\right)^2 + \frac{(ac - b^2)}{a}x_2^2. \end{aligned} \tag{8}$$

If both coefficients, a and $(ac - b^2)/a$ in (8) are positive, then Q will never be negative. It will equal zero only when

$$x_1 + \frac{b}{a}x_2 = 0 \quad \text{and} \quad x_2 = 0,$$

that is, when $x_1 = 0$ and $x_2 = 0$. In other words, if

$$|a| > 0 \quad \text{and} \quad \det A = \begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0,$$

then Q is positive definite. Conversely, in order for Q to be positive definite, we need both a and $\det A = ac - b^2$ to be positive.

Similarly, Q will be negative definite if and only if both coefficients in expression (8), a and $(ac - b^2)/a$, are negative. This situation occurs if and only if $a < 0$ and $ac - b^2 > 0$, that is, when the leading principal minors alternate in sign.

If $(ac - b^2)$, the second order principal minor, is negative, then the two coefficients in (8) have opposite signs. In particular,

$$Q\left(\frac{b}{a}, -1\right) = \frac{ac - b^2}{a} \quad \text{and} \quad Q(1, 0) = a$$

will have opposite signs; so Q is *indefinite*.

Example 16.4 Consider

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}.$$

Since $|A_1| = 2$ and $|A_2| = 5$, A is positive definite. Since $|B_1| = 2$ and $|B_2| = -2$, B is indefinite.

Example 16.5 Consider

$$C = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

Note that $|C_1| = 0$ and $|C_2| = 0$. The definiteness of C depends completely on c; C is positive semidefinite if $c \geq 0$ and negative semidefinite if $c \leq 0$. This is especially obvious if one looks at the corresponding quadratic form $Q_C(x_1, x_2) = cx_2^2$.

EXERCISES

16.1 Determine the definiteness of the following symmetric matrices:

$$a)\ \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad b)\ \begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix} \quad c)\ \begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix} \quad d)\ \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$$

$$e)\ \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix} \quad f)\ \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad g)\ \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$$

16.2 Let $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form on $\mathbf{R}^n$. By evaluating Q on each of the coordinate axes in $\mathbf{R}^n$, prove that a necessary condition for a symmetric matrix to be positive definite (positive semidefinite) is that all the diagonal entries be positive (nonnegative). State and prove the corresponding result for negative and negative semidefinite matrices. Give an example to show that this necessary condition is not sufficient.

16.3 Using the method of the previous exercise, sketch a proof that if A is positive (or negative) definite, then every principal submatrix of A is also positive (or negative) definite.

16.4 How many kth order principal minors will an $n \times n$ matrix have for each $k \leq n$?

16.5 Mimic the computation in (8) to prove Theorem 16.1 for a general symmetric 3×3 matrix. [Hint: After "completing the square" twice, you should find that

$$(x_1 \quad x_2 \quad x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

$$|A_1|\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3\right)^2 + \frac{|A_2|}{|A_1|}\left(x_2 + \frac{a_{11}a_{23} - a_{12}a_{13}}{|A_2|}x_3\right)^2 + \frac{|A_3|}{|A_2|}(x_3)^2.]$$

16.3 LINEAR CONSTRAINTS AND BORDERED MATRICES

Definiteness and Optimality

Keep in mind the fact that determining the definiteness of a quadratic form Q is equivalent to determining whether $\mathbf{x} = \mathbf{0}$ is a max, a min, or neither for the real-valued function Q. For example, $\mathbf{x} = \mathbf{0}$ is the unique *global minimum* of quadratic form Q if and only if Q is positive definite, by the very definition of positive definiteness. Similarly, $\mathbf{x} = \mathbf{0}$ is the unique *global maximum* of Q if and only if Q is negative definite.

The characterization of definiteness in Theorem 16.1 works only if there are no constraints in the problem under consideration, that is, if $\mathbf{x}$ can take on any value in $\mathbf{R}^n$. If there are constraints, the analysis becomes more delicate.

Example 16.6 The quadratic form $Q(x_1, x_2) = x_1^2 - x_2^2$ on $\mathbf{R}^2$ is indefinite; the origin is neither a max nor a min. But, if we restrict our attention to the x_1-axis, that is, if we impose the constraint $x_2 = 0$, then $Q(x_1, 0) = x_1^2$ has a strict global minimum at $x_1 = 0$, and therefore Q is positive definite on the constraint set $\{x_2 = 0\}$. Alternatively, if we impose the constraint $x_1 = 0$ and consider Q only on the x_2-axis, then $x_2 = 0$ is a global max of $Q(0, x_2) = -x_2^2$ and Q is negative definite on the subspace $\{x_1 = 0\}$. On the line $x_1 - 2x_2 = 0$, $Q(2x_2, x_2) = (2x_2)^2 - x_2^2 = 3x_2^2$ is positive definite.

As will be shown in Chapter 19, the second order condition which distinguishes maxima from minima in a *constrained* optimization problem is a condition on the definiteness of a quadratic form which is restricted to a linear subspace. Since most optimization problems in economics involve constraints on the variables under study, the rest of this chapter will discuss the definiteness of quadratic forms which are restricted to linear subspaces of $\mathbf{R}^n$.

Let us look in detail at the simplest such problem: the problem of determining the definiteness of, or of optimizing, a general quadratic form of two variables:

$$Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{9}$$

on the general linear subspace

$$Ax_1 + Bx_2 = 0. \tag{10}$$

In Example 16.6, we worked with $A = 0$, then with $B = 0$, and finally with $A = 1$ and $B = -2$.

Since our focus is on the matrix and not on the quadratic form itself, we have multiplied the coefficient of x_1x_2 in (9) by 2 so that we do not have to deal with $1/2$s in the corresponding matrix. The simplest approach to this problem is to solve (10) for x_1 in terms of x_2; obtain $x_1 = -Bx_2/A$, and then substitute this expression for x_1 in the objective function (9):

$$\begin{aligned} Q\left(-\frac{Bx_2}{A}, x_2\right) &= a\left(-\frac{Bx_2}{A}\right)^2 + 2b\left(-\frac{Bx_2}{A}\right)x_2 + cx_2^2 \\ &= \frac{aB^2}{A^2}x_2^2 - \frac{2bB}{A}x_2^2 + cx_2^2 \\ &= \frac{aB^2 - 2bAB + cA^2}{A^2}\, x_2^2. \end{aligned} \tag{11}$$

We conclude from (11) that Q is positive definite on the constraint set (10) if and only if $aB^2 - 2bAB + cA^2 > 0$ and negative definite on (10) if and only if $aB^2 - 2bAB + cA^2 < 0$. There is a convenient way of writing this expression:

$$aB^2 - 2bAB + cA^2 = -\det\begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix}. \tag{12}$$

The matrix in (12) is obtained by "bordering" the 2×2 matrix (9) of the quadratic Q on the top and left by the coefficients of the linear constraint (10). The following theorem summarizes these calculations.

Theorem 16.3 The quadratic form $Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ is positive (respectively, negative) definite on the constraint set $Ax_1 + Bx_2 = 0$ if and only if

$$\det\begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix}$$

is negative (respectively, positive).

This same result holds for the general problem of determining the definiteness of

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = (x_1 \quad \cdots \quad x_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{13}$$

on the linear constraint set

$$\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{14}$$

Border the matrix (13) of the quadratic form on the top and on the left by the matrix (14) of the linear constraints:

$$H = \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & B_{m1} & \cdots & B_{mn} \\ \hline B_{11} & \cdots & B_{m1} & a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{1n} & \cdots & B_{mn} & a_{1n} & \cdots & a_{nn} \end{array}\right). \tag{15}$$

We first need to figure out which submatrices of H to consider. In studying the quadratic (9) on the constraint set (10), we had only one condition to check in Theorem 16.3 because our single constraint in $\mathbf{R}^2$ resulted in a *one*-dimensional problem. The above problem (13, 14) has m linear equations of n variables. We therefore expect that this problem is really $n - m$ dimensional and therefore that we will have $n - m$ conditions to check for the matrix H in (15). Furthermore, from our experience in Section 16.2, we expect that we will look for leading principal minors of the *same* sign to check for positive definiteness and for leading principal minors of *alternating* signs to check for negative definiteness. The following theorem indicates that these expectations are correct, and it makes precise the exact sign patterns that we need to verify.

Theorem 16.4 To determine the definiteness of a quadratic form (13) of n variables, $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, when restricted to a constraint set (14) given by m linear equations $B\mathbf{x} = \mathbf{0}$, construct the $(n + m) \times (n + m)$ symmetric matrix H by bordering the matrix A above and to the left by the coefficients B of the linear constraints:

$$H = \begin{pmatrix} \mathbf{0} & B \\ B^T & A \end{pmatrix}.$$

Check the signs of the *last* $n - m$ leading principal minors of H, starting with the determinant of H itself.

(*a*) If $\det H$ has the same sign as $(-1)^n$ and if these last $n - m$ leading principal minors *alternate* in sign, then Q is *negative definite* on the constraint set $B\mathbf{x} = \mathbf{0}$, and $\mathbf{x} = \mathbf{0}$ is a strict global max of Q on this constraint set.

(*b*) If $\det H$ and these last $n - m$ leading principal minors all have the *same* sign as $(-1)^m$, then Q is *positive definite* on the constraint set $B\mathbf{x} = \mathbf{0}$, and $\mathbf{x} = \mathbf{0}$ is a strict global min of Q on this constraint set.

(*c*) If both of these conditions *a*) and *b*) are violated by *nonzero* leading principal minors, then Q is *indefinite* on the constraint set $B\mathbf{x} = \mathbf{0}$, and $\mathbf{x} = \mathbf{0}$ is neither a max nor a min of Q on this constraint set.

We will not present the rather intricate proof of this theorem here. Notice that its conclusions are consistent with the conclusions of Theorem 16.3 for $n = 2$ and $m = 1$, where for the case of two variables and one constraint, we only needed to compute $n - m = 2 - 1 = 1$ determinant. Exercise 16.9 looks at some important special cases of Theorem 16.4.

Example 16.7 To check the definiteness of

$$Q(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 + x_3^2 + x_4^2 + 4x_2x_3 - 2x_1x_4$$

on the constraint set

$$x_2 + x_3 + x_4 = 0, \qquad x_1 - 9x_2 + x_4 = 0,$$

form the bordered matrix

$$H_6 = \left(\begin{array}{cc|cccc} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -9 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & -9 & 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right).$$

Since this problem has $n = 4$ variables and $m = 2$ constraints, we need to check the largest $n - m = 2$ leading principal submatrices of H_6: H_6 itself and

$$H_5 = \left(\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -9 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 \\ 1 & -9 & 0 & -1 & 2 \\ 1 & 0 & 0 & 2 & 1 \end{array}\right).$$

Since $m = 2$ and $(-1)^2 = +1$, we need $\det H_6 > 0$ and $\det H_5 > 0$ to verify positive definiteness. Since $n = 4$ and $(-1)^n = +1$, we need $\det H_6 > 0$ and $\det H_5 < 0$ to verify negative definiteness. In fact, $\det H_6 = 24$ and $\det H_5 = 77$; so Q is positive definite on the constraint set, and $\mathbf{x} = \mathbf{0}$ is a min of Q restricted to the constraint set.

Remark If the test for constrained definiteness of Theorem 16.4 fails only because one or more of the last $n - m$ leading principal minors is zero, then we would like a test for semidefiniteness, analogous to the statement of Theorem 16.2 for the unconstrained problem. Unfortunately, tests for constrained semidefiniteness are much more tedious to state than the criteria described in Theorem 16.2. Fortunately, such tests are rarely required in applications.

One Constraint

Constrained maximization problems with just one effective constraint are common in economic theory. For the problem of checking the definiteness of a quadratic Q subject to a single constraint $A_1x_1 + \ldots + A_nx_n = 0$, Theorem 16.4 states that one needs to check the last $n - 1$ leading principal minors of

$$H_{n+1} = \begin{pmatrix} 0 & A_1 & \cdots & A_n \\ A_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ A_n & a_{1n} & \cdots & a_{nn} \end{pmatrix}. \tag{16}$$

The only omitted leading principal submatrices are:

$$H_1 = (0) \quad \text{and} \quad H_2 = \begin{pmatrix} 0 & A_1 \\ A_1 & a_{11} \end{pmatrix}.$$

Let's suppose that $A_1 \neq 0$. (One of the A_i's must be nonzero.) Then, $\det H_2 = -A_1^2 < 0$. Since $m = 1$ and $(-1)^1 = -1$, the criterion for constrained positive definiteness is that the last $n - 1$ leading principal minors of (16) are negative. Since $\det H_2 < 0$, this criterion is equivalent to the statement that the last n leading principal minors have the same sign. The criterion for constrained negative definiteness is that $\det H_{n+1}$ have the sign of $(-1)^n$ and that $\det H_3, \ldots, \det H_{n+1}$ alternate in sign. This means, in this case, that $\det H_3$ must be positive. It follows that the condition for constrained negative definiteness is equivalent to the condition that the last n leading principal minors of H_{n+1} alternate in sign. The following theorem summarizes this discussion for $m = 1$ and yields an easier-to-remember approach for the problem of determining definiteness when there is only one linear constraint.

Theorem 16.5 To determine the definiteness of a quadratic $Q(x_1, \ldots, x_n)$ subject to *one* linear constraint, form the usual $(n + 1) \times (n + 1)$ bordered matrix H, as in (16). Suppose that $A_1 \neq 0$. If the last n leading principal minors of H_{n+1} have the same sign, Q is positive definite on the constraint set (and $\mathbf{x} = \mathbf{0}$ is a constrained min of Q). If the last n leading principal minors of H_{n+1} alternate in sign, Q is negative definite on the constraint set (and $\mathbf{x} = \mathbf{0}$ is a constrained max of Q).

Other Approaches

For the sake of completeness, we mention two alternative approaches to the problem of determining the definiteness of a quadratic form of n variables subject to m linear equations. The statement of Theorem 16.4 focuses on the sign of the largest submatrix H_{m+n} as the cornerstone of the algorithm. Some presentations focus instead on the smallest of the last $n - m$ leading principal submatrices: H_{2m+1}, the $(2m + 1)$th order leading principal submatrix. Theorem 16.4 implies the following alternative checks:

(A) To verify positive definiteness, check that $\det H_{2m+1}$ has the same sign as $(-1)^m$ and that all the larger leading principal minors have this sign too.

(B) To verify negative definiteness, check that $\det H_{2m+1}$ has the sign of $(-1)^{m+1}$ and that the leading principal minors of larger order alternate in sign.

Some texts prefer to construct the bordered matrix H by bordering the matrix A of the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ *below* and to the *right* by the matrix B for the linear equations $B\mathbf{x} = \mathbf{0}$:

$$H_{m+n} = \begin{pmatrix} A & B^T \\ B & \mathbf{0} \end{pmatrix}.$$

In this situation, one must still check $n - m$ principal minors. However, the corresponding principal submatrices are no longer leading ones, but "border-preserving" ones. One removes from H_{m+n}, one at a time, the $n - m - 1$ rows and columns which contain the last $n - m - 1$ rows and columns of the matrix A, that is, rows and columns $n, n - 1, \ldots, m + 2$ of H_{m+n}.

Example 16.8 To use this approach for the problem in Example 16.7, form the bordered matrix

$$\hat{H} = \left(\begin{array}{cccc|cc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & -9 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & -9 & 0 & 1 & 0 & 0 \end{array}\right)$$

and then form the submatrix $\hat{H}_5$ by removing row 4 and column 4 from $\hat{H}$, the row and column just before the border of $\hat{H}$:

$$\hat{H}_5 = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 2 & 1 & -9 \\ 0 & 2 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 \\ 1 & -9 & 0 & 0 & 0 \end{array}\right).$$

Note that $\det \hat{H} = 24$ and $\det \hat{H}_5 = 77$, just as we found for the corresponding minors in Example 16.7.

EXERCISES

16.6 Determine the definiteness of the following constrained quadratics:

a) $Q(x_1, x_2) = x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.

b) $Q(x_1, x_2) = 4x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.

c) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$, subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$.

d) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_3 - 2x_1x_2$, subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$.

e) $Q(x_1, x_2, x_3) = x_1^2 - x_3^2 + 4x_1x_2 - 6x_2x_3$, subject to $x_1 + x_2 - x_3 = 0$.

16.7 Prove that statements A and B above are equivalent to the statement of Theorem 16.4.

16.8 Use the theory of determinants to show why the corresponding minors in Examples 16.7 and 16.8 have the same values.

16.9 Use the techniques of Theorem 16.3 to verify Theorem 16.4 for the general problem with: *a*) three variables and one constraint, *b*) three variables and two constraints.

16.4 APPENDIX

This section presents the proof of Theorem 16.1. This proof has two major ingredients: the principle of induction and the theory of partitioned matrices as developed in Section 8.7. We will prove Theorem 16.1 for positive definite matrices and leave the proof for negative definite matrices as an exercise. First, we need two simple lemmas.

Lemma 16.1 If A is a positive or negative definite matrix, then A is nonsingular.

Proof Suppose that such an A is singular. Then, there exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. But then,

$$\mathbf{x}^T \cdot A\mathbf{x} = \mathbf{x}^T \cdot \mathbf{0} = 0,$$

a contradiction to the definiteness of A. ■

Lemma 16.2 Suppose that A is a symmetric matrix and that Q is a nonsingular matrix. Then, Q^TAQ is a symmetric matrix, and A is positive (negative) definite if and only if Q^TAQ is positive (negative) definite.

Proof To see that Q^TAQ is symmetric, one checks directly that it equals its own transpose:

$$(Q^TAQ)^T = Q^TA^T(Q^T)^T = Q^TA^TQ = Q^TAQ.$$

Suppose that Q^TAQ is positive definite. Let $\mathbf{x} \neq \mathbf{0}$ be an arbitrary nonzero vector in $\mathbf{R^n}$. Since Q is nonsingular, there exists a nonzero vector $\mathbf{y}$ such that $\mathbf{x} = Q\mathbf{y}$. Then

$$\mathbf{x}^TA\mathbf{x} = (Q\mathbf{y})^TA(Q\mathbf{y}) = \mathbf{y}^TQ^TAQ\mathbf{y} = \mathbf{y}^T(Q^TAQ)\mathbf{y},$$

which is positive, since Q^TAQ is positive definite. Therefore, A is positive definite.

On the other hand, if A is positive definite and $\mathbf{z}$ is an arbitrary nonzero vector, then $Q\mathbf{z}$ will be nonzero too, since Q is nonsingular. Therefore,

$$0 < (Q\mathbf{z})^TA(Q\mathbf{z}) = \mathbf{z}^TQ^TAQ\mathbf{z} = \mathbf{z}^T(Q^TAQ)\mathbf{z},$$

and Q^TAQ is positive definite. ■

Theorem 16.1 Let A be a symmetric matrix. Then, A is positive definite if and only if all its leading principal minors are positive.

Proof We will prove this result by using induction on the size n of A. The result is trivially true for 1×1 matrices. We proved it for 2×2 matrices in Section 16.2. We suppose that the theorem is true for $n \times n$ matrices and prove it true for $(n+1) \times (n+1)$ matrices.

Let A be an $(n+1) \times (n+1)$ symmetric matrix. Write A_j for the $j \times j$ leading principal submatrix of A for $j = 1, \ldots, n+1$.

We first prove that if all the A_j's have positive determinants, then A is positive definite. The leading principal submatrices of A_n are $A_1, \ldots, A_n$, which are positive definite by hypothesis, since they are the first n leading principal submatrices of A. By the inductive hypothesis that the theorem is true for $n \times n$ matrices, the $n \times n$ symmetric matrix A_n is positive definite. By Lemma 16.1 above, A_n is invertible. Partition A as

$$A = \left(\begin{array}{c|c} A_n & \mathbf{a} \\ \hline \mathbf{a}^T & a_{n+1,n+1} \end{array}\right), \tag{17}$$

where $\mathbf{a}$ denotes the $n \times 1$ column matrix

$$\mathbf{a} = \begin{pmatrix} a_{1,n+1} \\ \vdots \\ a_{n,n+1} \end{pmatrix}.$$

Let $d = a_{n+1,n+1} - \mathbf{a}^T(A_n)^{-1}\mathbf{a}$, let I_n denote the $n \times n$ identity matrix, and let $\mathbf{0}_n$ denote the $n \times 1$ column matrix of all 0s. Then, the matrix A in (17) can be written as

$$\begin{aligned} A &= \left(\begin{array}{c|c} I_n & \mathbf{0}_n \\ \hline (A_n^{-1}\mathbf{a})^T & 1 \end{array}\right) \left(\begin{array}{c|c} A_n & \mathbf{0}_n \\ \hline \mathbf{0}_n^T & d \end{array}\right) \left(\begin{array}{c|c} I_n & A_n^{-1}\mathbf{a} \\ \hline \mathbf{0}_n^T & 1 \end{array}\right) \\ &\equiv Q^T B Q. \end{aligned} \tag{18}$$

(Exercise.) By properties of the determinant,

$$\det Q = \det Q^T = 1 \qquad \text{and} \qquad \det B = d \cdot \det A_n.$$

Therefore,
$$\det A = d \cdot \det A_n. \tag{19}$$

Since $\det A > 0$ and $\det A_n > 0$, then $d > 0$.

Let $\mathbf{X}$ be an arbitrary $(n+1)$-vector. Write $\mathbf{X}$ as

$$\mathbf{X} = \begin{pmatrix} \mathbf{x} \\ x_{n+1} \end{pmatrix},$$

where $\mathbf{x}$ is an n-vector. Then,

$$\mathbf{X}^T B \mathbf{X} = (\mathbf{x}^T \quad x_{n+1}) \left(\begin{array}{c|c} A_n & \mathbf{0}_n \\ \hline \mathbf{0}_n^T & d \end{array} \right) \begin{pmatrix} \mathbf{x} \\ x_{n+1} \end{pmatrix} \tag{20}$$
$$= \mathbf{x}^T A_n \mathbf{x} + d x_{n+1}^2.$$

Since A_n is positive definite by inductive hypothesis and $d > 0$, this last expression is strictly positive. Therefore, $B = Q^T A Q$ is positive definite. By Lemma 16.2, A is positive definite.

To prove the converse — A positive definite implies that all the $|A_j|$'s are positive — we use induction once more. We have seen that this result is true for 1×1 and 2×2 matrices. Assume that it is true for $n \times n$ symmetric matrices, and let A be an $(n+1) \times (n+1)$ positive definite symmetric matrix. We first show that all the A_j's are positive definite. Let $\mathbf{x}_j$ be a nonzero j-vector, and let $\mathbf{0}^*$ be the zero $(n+1) - j$ vector. Then

$$0 < (\mathbf{x}_j^T \quad \mathbf{0}^*) A \begin{pmatrix} \mathbf{x}_j \\ \mathbf{0}^* \end{pmatrix} = \mathbf{x}_j^T A_j \mathbf{x}_j;$$

and A_j is positive definite.

In particular, since A_n is positive definite, the inductive hypothesis tells us that $A_1, \ldots, A_n$ all have positive determinants. We need only prove that the determinant of A itself is positive. Since A_n is invertible, we can once again write A as $Q^T B Q$ as in (18) and conclude that (19) still holds. Since A is positive definite, B is positive definite by Lemma 16.2. Choose $\mathbf{X}$ in (20) so that $\mathbf{x} = \mathbf{0}$ and $x_{n+1} = 1$. Then,

$$0 < \mathbf{X}^T B \mathbf{X} = d.$$

Since $\det A_n > 0$ and $d > 0$, $\det A > 0$. ■

EXERCISES

16.10 Show that the block decomposition (18) is correct.

16.11 Prove the corresponding theorem for negative definite matrices.