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CAPITAL-LABOR SUBSTITUTION AND ECONOMIC EFFICIENCY[1]

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IN many branches of economic theory, it is necessary to make some assumption about the extent to which capital and labor are substitutable for each other. In the absence of empirical generalizations about this phenomenon, theorists have chosen simple hypotheses, which have become widely accepted through frequent repetition. Two competing alternatives hold the field at present: the Walras-Leontief-Harrod-Domar assumption of constant input coefficients;[2] and the Cobb-Douglas function, which implies a unitary elasticity of substitution between labor and capital. From a mathematical point of view, zero and one are perhaps the most convenient alternatives for this elasticity. Economic analysis based on these assumptions, however, often leads to conclusions that are unduly restrictive.

The crucial nature of the substitution assumption can be illustrated in various fields of economic theory:

(i) The unstable balance of the Harrod-Domar model of growth depends in a critical way on the asumption of zero substitution between labor and capital, as Solow [15], Swan [18], and others have shown.

(ii) The effects of varying factor endowments on international trade hinge on the shape of particular production functions. In this case, either zero or unitary elasticities of substitution in *all* sectors of the economy lead to Samuelson's strong assumption as to the invariability of the ranking of factor proportions. Variations in elasticity among sectors imply reversals of factor intensities at different factor prices with quite different consequences for trade and factor returns.

(iii) In analyzing the relative shares of income received by the factors of production, it is tempting to assume unit elasticity of substitution to agree with the supposed constancy of the labor share in the United States. Recent work has called into question both the observed constancy and the necessity of the assumption [9, 17].

Turning to empirical evidence, we find every indication of varying degrees of substitutability in different types of production. Technological alternatives are numerous and flexible in some sectors, limited in others; and uniform substitutability is most unlikely. The difference in elasticities is confirmed by direct observation of capital-labor proportions, which show much more variation among countries in some sectors than in others.

The starting point for the present study was the empirical observation that the value added per unit of labor used within a given industry varies across countries with the wage rate. Evidence of this relationship for 24 manufacturing industries in a sample of 19 countries is given in section I. A regression of the labor productivity on the wage rate shows a highly significant correlation in all industries and also a considerable variation in the regression coefficients.

These empirical findings led to attempts to derive a mathematical function having the properties of (i) homogeneity, (ii) constant elasticity of substitution between capital and labor, and (iii) the possibility of different elasticities for

[1] This study grows out of the research program of the Stanford Project for Quantitative Research in Economic Development. It is one in a series of analyses based on international comparisons of the economic structure. Hendrik Houthakker contributed substantially to the formulation of both the statistical and theoretical analyses. Arrow's participation was aided by Contract 251(33), Task N4047–004, Office of Naval Research.

[2] It is only fair to note that the general equilibrium theories of Walras and Leontief never assume fixed proportions for gross aggregates like capital and labor.

different industries. In section II it is shown that there is one general production function having these properties; it includes the Leontief and Cobb-Douglas functions as special cases.[3] The function contains three parameters, which are identified as the substitution parameter, the distribution parameter, and the efficiency parameter.

To test the validity of this formulation, we examine in section III the fragmentary information available on direct use of capital and also the deviations from the regression analysis. These tests, while inconclusive, suggest the working hypothesis that the efficiency parameter varies from country to country but that the other two are constants for each industry.

In this form, the constant-elasticity-of-substitution (CES) production function implies a number of predictable differences in the structure of production and trade between countries having different relative factor costs. Some of these are investigated in section IV through a comparison between Japan and the United States of factor use and relative prices. The results indicate the extent of substitutability between labor and capital in all sectors of the economy and also support the hypothesis of varying efficiency given in section III.

Finally, the CES production function is applied in section V to a time-series analysis of all non-farm production in the United States. The results show an over-all elasticity of substitution between capital and labor significantly less than unity and provide a further test of the validity of the production function itself.

I. Variation in Labor Inputs with Labor Cost

International comparisons are probably the best available source of information on the effect of varying factor costs on factor inputs. The observed range of variation in the relative costs of labor and capital is of the order of 30:1, which is much greater than that observed in a single country over any period for which data are available. The observations on factor inputs refer to a specific industry or set of technological operations rather than to the different industries conventionally employed in cross-section studies within a single country. Finally, taking observations that are close together in time, one can assume access to approximately the same body of technological knowledge; while not strictly true, this hypothesis is more valid than the same assumption applied to time-series analysis.

[3] This function and its properties were arrived at independently by Solow and Arrow.

A. *Data*

The substantial number of industrial censuses in the postwar period that use comparable industrial classifications makes it possible to exploit some of these potential advantages of inter-country analysis. The sample used, the data collected, and the relationships explored are determined primarily by the nature of the census materials.

Countries in the sample. The sample consists of countries having the requisite wage and output data in a reasonable number of industries. The countries, average wage rates, and number of industries available for each are shown in Table 1. The data pertain to different years between 1949 and 1955.

TABLE 1. — COUNTRIES IN THE SAMPLE

Country	Year of Census	Average wage [a] (*Current dollars*)	Number of industries used [b]
1. United States	1954	3841	24
2. Canada	1954	3226	23
3. New Zealand	1955/56	1980	22
4. Australia	1955/56	1926	24
5. Denmark	1954	1455	24
6. Norway	1954	1393	22
7. Puerto Rico	1952	1182	17
8. United Kingdom	1951	1059	24
9. Colombia	1953	924	24
10. Ireland	1953	900	15
11. Mexico	1951	524	21
12. Argentina	1950	519	24
13. Japan	1953	476	23
14. El Salvador	1951	445	16
15. Brazil	1949	436	10
16. S. Rhodesia	1952	384	6
17. Ceylon	1952	261	11
18. India	1953	241	17
19. Iraq	1954	213	2

[a] Unweighted average of wages in industries in sample.
[b] Industry data are given in the appendix.

Industries. Data were collected for all industries at the three-digit level of the United Nations International Standard Industrial Classification having sufficient observations (at least 10). The 24 industries analyzed are listed in Table 2 and defined in the ISIC. There is, of course, considerable variation in the composition of output within a given industrial category among countries at different income levels, which cannot be allowed for here.

Labor inputs and costs. Labor inputs are measured in man-years per $1000 of value added. They include production workers, salaried employees, and working proprietors. Labor costs are measured by the average annual wage payment, computed as the total wage bill divided by the number of employees. The data on wage payments for different countries include varying proportions of non-wage benefits, and we made no allowance for such variations. The data on employment are not corrected for inter-country differences in the number of hours worked per year or the age and sex composition of the labor force. The data for each industry are given in the appendix.

Exchange rates. All conversions from local currency values into U.S. dollars were at official exchange rates or at free market rates where multiple exchange rates prevailed. No allowance was made for the variation in the purchasing power of the dollar between different census years.

Capital inputs. Data on capital inputs or rates of return are available only for a small number of countries and industries. They are therefore omitted from the initial statistical analysis and utilized in section III to test the validity of the production function that is proposed in section II.

B. *Regression Analysis*

The variables available for statistical analysis are as follows:

V : value added in thousands of U.S. dollars

L : labor input in man-years

W : money wage rate (total labor cost divided by L) in dollars per man-year.

As an aid in formulating the regression analysis, we make the following preliminary assumptions, the validity of which will be examined in section III.

(1) Prices of products and material inputs do not vary systematically with the wage level.

(2) Overvaluation or undervaluation of exchange rates is not related to the wage level.

(3) Variation in average plant size does not affect the factor inputs.

TABLE 2. — RESULTS OF REGRESSION ANALYSIS[a]

ISIC No.	Industry	Regression equations: Log a	b	Standard error S_b	Coeff. of deter. $\bar{R}^2$	Test of Significance on b: Degrees of freedom	Confidence level for b different from 1
202	Dairy products	.419	.721	.073	.921	14	99%
203	Fruit and vegetable canning	.355	.855	.075	.910	12	90
205	Grain and mill products	.429	.909	.096	.855	14	*
206	Bakery products	.304	.900	.065	.927	14	80
207	Sugar	.431	.781	.115	.790	11	90
220	Tobacco	.564	.753	.151	.629	13	80
231	Textile — spinning and weaving	.296	.809	.068	.892	16	98
232	Knitting mills	.270	.785	.064	.915	13	99
250	Lumber and wood	.279	.860	.066	.910	16	95
260	Furniture	.226	.894	.042	.952	14	95
271	Pulp and paper	.478	.965	.101	.858	14	*
280	Printing and publishing	.284	.868	.056	.940	14	95
291	Leather finishing	.292	.857	.062	.921	12	95
311	Basic chemicals	.460	.831	.070	.898	14	95
312	Fats and oils	.515	.839	.090	.869	12	90
319	Miscellaneous chemicals	.483	.895	.059	.938	14	90
331	Clay products	.273	.919	.098	.878	11	*
332	Glass	.285	.999	.084	.921	11	*
333	Ceramics	.210	.901	.044	.974	10	95
334	Cement	.560	.920	.149	.770	10	*
341	Iron and steel	.363	.811	.051	.936	11	99
342	Non-ferrous metals	.370	1.011	.120	.886	8	*
350	Metal products	.301	.902	.088	.897	11	*
370	Electric machinery	.344	.870	.118	.804	12	*

[a] From data given in the Appendix.
* Not significant at 80% or higher levels of confidence.

(4) The same technological alternatives are available to all countries.

On these assumptions, we can treat $1000 of value added as a unit of physical output in each industry. We also assume a single production function for all countries, which implies that there will be a determinate relation between the labor input per unit of value added and the wage rate. Before exploring the possible forms of this function in detail, we tested two simple relations among the three variables statistically:

$$\frac{V}{L} = c + dW + \eta \tag{1a}$$

$$\log \frac{V}{L} = \log a + b \log W + \epsilon. \tag{1b}$$

Both functions give good fits to the observations, the logarithmic form being somewhat better. The results of the latter regression are shown in Table 2.[4] It is apparent from the small standard errors of b and the high coefficient of determination $\bar{R}^2$ that the fit is relatively good. In 20 out of 24 industries, over 85 per cent of the variation in labor productivity is explained by variation in wage rates alone.[5]

C. *Implied Properties of the Production Function*

The regression analysis provides an important basis for the derivation of a more general production function: the finding that a linear logarithmic function provides a good fit to the observations of wages and labor inputs. The theoretical analysis of the next section will therefore start from this assumption.

It is shown in section II that under the assumptions made here the coefficient b is equal to the elasticity of substitution between labor and capital. It is therefore of interest to determine the number of industries in which the elasticity is significantly different from 0 or 1, the values most commonly assumed for it. Results of a t test of the second hypothesis are given in Table 2. In all cases, the value of b is significantly different from zero at a 90 per cent level of confidence. In 14 out of 24 industries it is significantly different from 1 at 90 per cent or higher levels of confidence. We therefore reject these hypotheses as inadequate descriptions of the possibilities for combining labor and capital, and we proceed to derive a production function that allows for a different elasticity in each industry.

[4] Independently of this study, J. B. Minasian has fitted equation (1b) to U. S. interstate data for a number of industries in "Elasticities of Substitution and Constant-Output Demand Curves for Labor," *Journal of Political Economy*, June 1961, 261–270. (Note added in proof.)

[5] For the economy as a whole, the level of wages depends on the level of labor productivity, but for a given industry the labor input per unit of output is adjusted to the prevailing wage level in the country with relatively small deviations due to the relative profitability of the given industry.

II. A New Class of Production Functions

Section I presents observations on the relation between V/L and w within each of several industries at a single point of time. It is a natural first step to give an account of the results in terms of profit-maximizing responses to given factor prices. Under the assumptions of constant returns to scale and competitive labor markets, the standard theory of production shows how any particular production function entails a particular relation between V/L and w. We shall show that the reverse implication also holds: that a particular relation between V/L and w determines the corresponding production function up to one arbitrary constant.

A. *Output per unit of Labor, Real Wages, and the Production Function under Constant Returns*

If the production function in a particular industry is written $V = F(K,L)$, and assumed to be homogeneous of degree one, then $V/L = F(K/L, 1)$; and if we put $V/L = y, K/L = x$, we can say $y = f(x)$. In these terms the marginal products of capital and labor are $f'(x)$ and $f(x) - xf'(x)$ respectively. Let w be the wage rate with output as *numéraire*. If the labor and product markets are competitive then

$$w = f(x) - xf'(x) \tag{2}$$

which can be inverted to give a functional relation between x and w, and thence, since $y=f(x)$, a monotone increasing relation between y and w. Conversely, suppose we begin (as we do) with such an observed relation between y and w, say $y = \phi(w)$. Then from (1) we see that

$$y = \phi(y - x\frac{dy}{dx}) \tag{3}$$

which is a differential equation for $y(x)$. It will have a solution

$$y = f(x; A) \tag{4a}$$

where A is a constant of integration. Returning to the original variables we get the one-parameter family of production functions

$$V = Lf(K/L; A). \tag{4b}$$

Of course for (4) to do duty as a production function it should have positive marginal productivities for both inputs and be subject to the usual diminishing returns when factor-proportions vary. An elementary calculation shows that these conditions are equivalent to requiring that $f'(x) > 0$ and $f''(x) < 0$. The latter condition is also sufficient to permit the inversion of (2). Geometrically these conditions state that output per unit of labor is an increasing function of the input of capital per unit of labor, convex from above, just as the curve is normally drawn. In addition one would desire that $f(x) > 0$ for $x > 0$. All these requirements should hold for at least some value of A.

This way of generating production functions brings to light a connection with the elasticity of substitution which does not seem to have been noticed in the literature, although closely related results were obtained by Hicks and others (see Allen [2], 373). The slight difference has to do with the treatment of product price. Let s stand for the marginal rate of substitution between K and L (the ratio of the marginal product of L to that of K). Then the elasticity of substitution σ is defined simply as the elasticity of K/L with respect to s, along an isoquant. For constant returns to scale it turns out,[6] in our notation,

$$\sigma = -\frac{f'(f - xf')}{xff''}. \tag{5}$$

Now consider the relation between y and w as determined implicitly by (2). Differentiating with respect to w we obtain

$$1 = f'\frac{dx}{dy}\frac{dy}{dw} - xf''\frac{dx}{dy}\frac{dy}{dw} - f'\frac{dx}{dy}\frac{dy}{dw}$$

and since $\dfrac{dx}{dy} = \dfrac{1}{f'}$,

$$\frac{dy}{dw} = -\frac{f'}{xf''}.$$

[6] On all this see Allen [2], 340–43.

Thus the elasticity of y with respect to w is, from (2),

$$\frac{w}{y}\frac{dy}{dw} = -\frac{f'(f - xf')}{xff''} = \sigma. \tag{6}$$

That is to say, if the relation between V/L and w arises from profit-maximization along a constant-returns-to-scale production function, the elasticity of the resulting curve is simply the elasticity of substitution. Information about σ can be obtained, under these assumptions, from observation of the joint variation of output per unit of labor and the real wage.

We may also observe another simple and interesting relation associated with production functions homogeneous of degree one. As has been seen the marginal productivity of capital is a decreasing function of x, the capital-labor ratio, while the marginal productivity of labor is of course an increasing function. Hence, for competitive markets, the gross rental, r, measured with output as *numéraire*, is a decreasing function of the wage rate. More specifically, we may differentiate the relations, $r = f'(x)$ and (2) to yield,

$$\frac{dr}{dx} = f''(x); \frac{dw}{dx} = f' - xf'' - f' = -xf''$$

so that

$$\frac{dr}{dw} = \left(\frac{dr}{dx}\right) \Big/ \left(\frac{dw}{dx}\right) = -\frac{1}{x} = -\frac{L}{K},$$

whence the elasticity of the rate of return with respect to the wage rate is,

$$\frac{w}{r}\frac{dr}{dw} = -\frac{wL}{rK}, \tag{7}$$

i.e., the ratio of labor's share to capital's share in value added.

B. *Rationalizing the Data of Section I*

We found in section I that in general a linear relationship between the logarithms of V/L and w, i.e.,

$$\log y = \log a + b \log w \tag{8}$$

gives a good fit. Along such a curve, the elasticity of y with respect to w is constant and equal to b. We are forewarned that the implied production function will have a constant elasticity of substitution equal to b, so that in deducing it we provide a substantial generalization of the Cobb-Douglas function. Indeed the

Cobb-Douglas family is the special case $b = 1$ in (8). Our empirical results imply that elasticities of substitution tend to be less than one, which contrasts strongly with the Cobb-Douglas view of the world. We will return subsequently to the distributional and other implications of this conclusion.

The differential equation (3) becomes

$$\log y = \log a + b \log \left(y - x\frac{dy}{dx}\right). \tag{9}$$

Taking antilogarithms and solving for $\frac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{a^{1/b}\, y - y^{1/b}}{a^{1/b}\, x} = \frac{y(1 - \alpha y^{\rho})}{x}$$

where we have set $\alpha = a^{-1/b}$ and $\rho = \frac{1}{b} - 1$ for convenience. The equation

$$\frac{dx}{x} = \frac{dy}{y(1 - \alpha y^{\rho})}$$

has a partial-fractions expansion:

$$\frac{dx}{x} = \frac{dy}{y} + \frac{\alpha y^{\rho-1}\, dy}{1 - \alpha y^{\rho}}$$

which can be integrated to yield

$$\log x = \log y - \frac{1}{\rho} \log (1 - \alpha y^{\rho}) + \frac{1}{\rho} \log \beta$$

or

$$x^{\rho} = \frac{\beta y^{\rho}}{1 - \alpha y^{\rho}}$$

which in turn can be solved for y^{ρ}, and then y, to give

$$y = x(\beta + \alpha x^{\rho})^{-1/\rho} = (\beta x^{-\rho} + \alpha)^{-1/\rho}. \tag{10}$$

Written out in full the production function is:

$$\begin{aligned} V &= L(\beta K^{-\rho} L^{\rho} + \alpha)^{-1/\rho} \\ &= (\beta K^{-\rho} + \alpha L^{-\rho})^{-1/\rho}. \end{aligned} \tag{11}$$

As for our requirements on the shape of the production function, it is clear that $y > 0$ for $x > 0$ as long as $\alpha > 0$ and $\beta > 0$. Differentiation of (10) shows that the only requirement for positive marginal productivities is $\beta > 0$. A second differentiation yields one further condition for diminishing returns, namely $\rho + 1 > 0$ which is equivalent to $b > 0$ and in accordance with our empirical results.

The family of production functions described by (10) or (11) comprises all those which exhibit a constant elasticity of substitution for all values of K/L. To be precise, the elasticity of substitution $\sigma = 1/(1+\rho) = b$. For this reason we will call (10) or (11) a constant-elasticity-of-substitution production function (abbreviated to CES).[7] Admissible values of ρ run from -1 to ∞, which permits σ to range from $+\infty$ to 0. Since our empirical values of b are almost all significantly less than one, they imply positive values of ρ and elasticities of substitution in different industries generally less than unity.

C. *Properties of the CES Production Function*

We can write (10) and (11) more symmetrically by setting $\alpha + \beta = \gamma^{-\rho}$ and $\beta\gamma^{\rho} = \delta$, in which notation they become

$$y = \gamma[\delta x^{-\rho} + (1-\delta)]^{-1/\rho} \tag{12}$$

$$V = \gamma[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-1/\rho}\ . \tag{13}$$

A change in the parameter γ changes the output for any given set of inputs in the same proportion. It will therefore be referred to as the (neutral) *efficiency parameter*. The parameter ρ, as has just been seen, is a transform of the elasticity of substitution and will be termed the *substitution parameter*. It will be seen below (equation 23) that for any given value of σ (equivalently, for any given value of ρ), the functional distribution of income is determined by δ, the *distribution parameter*.

Apart from the efficiency parameter (which can be made equal to one by appropriate choice of output units), (13) is a class of function known in the mathematical literature as a "mean value of order $-\rho$."[8]

The lowest admissible value for ρ is -1; this implies an infinite elasticity of substitution and therefore straight-line isoquants. One verifies this by putting $\rho = -1$ in (13).

For values of ρ between -1 and 0 we have elasticities of substitution greater than unity. From (12) we see that $y \to \infty$ as $x \to \infty$, and $y \to \gamma(1-\delta)^{-1/\rho}$ as $x \to 0$. That is to say, output per unit of labor becomes indefinitely large as the ratio of capital to labor increases; but as the capital/labor ratio approaches zero, the av-

[7] We note that Trevor Swan has independently deduced the constant-elasticity-of-substitution property of (11). The function itself was used by Solow [15], 77, as an illustration.

[8] See Hardy, Littlewood, and Pólya [7], 13. It may also be shown that the function (13) is the most general function which can be computed on a suitable slide rule.

erage product of labor approaches a positive lower limit.

The case $\rho = 0$ yields an elasticity of substitution of unity and should, therefore, lead back to the Cobb-Douglas function. This is not obvious from (13), since as $\rho \to 0$ the right-hand side is an indeterminate form of the type 1^∞. But in fact the limit *is* the Cobb-Douglas function. This can be seen (a) by direct application of L'Hôpital's Rule to (13); (b) by integration of (9) with $b = 1$; or (c) by appealing to the purely mathematical theorem that the mean value of order zero is the geometric mean.[9] Thus the limiting form of (13) at $\rho = 0$ is indeed $V = \gamma K^\delta L^{1-\delta}$.[10]

For $0 < \rho < \infty$, which is the empirically interesting case, we have $\sigma < 1$. The behavior is quite different from the case $-1 < \rho < 0$. As $x \to \infty$, $y \to \gamma(1-\delta)^{-1/\rho}$; as $x \to 0$, $y \to 0$. That is, as a fixed dose of labor is saturated with capital, the output per unit of labor reaches an upper limit. And as a fixed dose of capital is saturated with labor, the productivity of labor tends to zero.

Whenever $\rho > -1$, the isoquants have the right curvature ($\rho = -1$ is the case of straight-line isoquants, and $\rho < -1$ is ruled out precisely because the isoquants have the wrong curvature). The cases $\rho < 0$ and $\rho \geqslant 0$ are different; when $\rho < 0$, the isoquants intersect the K and L axes, while when $\rho \geqslant 0$, the isoquants only approach the axes asymptotically. Both cases are illustrated in Chart 1 of section IV.

Our survey of possible values of ρ concludes with two final remarks. The case $\rho = 1$, $\sigma = \frac{1}{2}$ is seen to be the ordinary harmonic mean. And as $\rho \to \infty$, the elasticity of substitution tends to zero and we approach the case of fixed proportions. We may prove this by making the appropriate limiting process on (13). And once again the general theory of mean values assures us that as a mean value of order $-\infty$ we have[11]

$$\lim_{\rho \to \infty} \gamma[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-1/\rho}$$
$$= \gamma \min(K,L) = \min\left(\frac{K}{\gamma^{-1}}, \frac{L}{\gamma^{-1}}\right) \quad (14)$$

This represents a system of right-angled isoquants with corners lying on a 45° line from the origin. But it is clearly more general than that, since the location of the corners can be changed simply measuring K and L in different units.

So far we have simply provided one possible rationalization of the data of section I. We turn next to some of the testable implications of the model, and in so doing we consider the possibility of lifting or at least testing the hypothesis of constant returns to scale. Further economic implications of the CES production function are discussed in section IV below.

D. *Testable Implications of the Model*

1. *Returns to scale.* So far we have assumed the existence of constant returns to scale. This is more than just convenience; it is at least suggested by the existence of a relationship between V/L and w, independent of the stock of capital. Indeed, homogeneity of degree one (together with competition in the labor and product markets) entails the existence of such a relationship. Clearly, not all production functions admit of a relationship between V/L and $w = \partial V/\partial L$; the class which does so, however, is somewhat broader than the homogeneous functions of degree one. We have the following precise result: if the labor and product markets are competitive, and if profit-maximizing behavior along a production function $V = F(K,L)$ leads to a functional relationship between w and V/L, then $F(K,L) = H(C(K),L)$ where H is homogeneous of degree one in C and L, and C is an increasing function of K.

In proof, since $w = \partial V/\partial L$, we can write this functional relation as:

$$\frac{\partial V}{\partial L} = h\left(\frac{V}{L}\right).$$

Since this holds independently of K we may hold K constant and proceed as with an ordinary differential equation. Introducing $y = V/L$, we have $L\partial y/\partial L + y = \partial V/\partial L$ and therefore $\partial y/\partial L = \dfrac{h(y) - y}{L}$. Since K is fixed we may write this

$$\frac{dy}{h(y) - y} = \frac{dL}{L}$$

and integrate to get

$$L = Cg(y) \quad (15)$$

[9] Hardy, Littlewood, and Pólya [7], 15, Theorem 3.

[10] This special case reinforces our singling-out of δ as a distribution parameter.

[11] Hardy, Littlewood, and Pólya [7], 15, Theorem 4.

where $g(y) = \exp \int^y \frac{dy}{h(y) - y}$; and C, the constant of integration must be taken as a function of K. (We also assume $h(y) < y$; that is, the average productivity of labor exceeds its marginal productivity.) Upon inversion of (15) we have y as a function of $L/C(K)$ alone, say $y = G(L/C(K))$ and therefore

$$V = LG\left(\frac{L}{C(K)}\right) = H(C(K), L) \tag{16}$$

as asserted, where H is homogeneous of degree one in its arguments. If K is to have positive marginal productivity, C must be an increasing function of K, since $g(y)$ is decreasing.

Thus under our assumptions production exhibits constant returns to scale, not necessarily in K and L, but in $C(K)$ and L. We have constant returns to scale if C is proportional to K. But $C(K)$ can be given an interpretation in any case. Let P represent all non-labor income, whether returns to capital or not. Then by Euler's Theorem, $P = C\,\partial H/\partial C$. And $\partial V/\partial K = (\partial H/\partial C)\,(dC/dK)$. Hence

$$\frac{C}{C'} = \frac{P}{\dfrac{\partial V}{\partial K}} \tag{17}$$

so that C/C' represents the "present value" of the stream of profits, discounted by the marginal productivity of capital.

The argument leading to (16) provides us with an empirical test of the hypothesis of constant returns to scale. As we have noted, the latter is equivalent to $C(K)/K$ being constant. But from (15),

$$\frac{C}{K} = \frac{L}{K}\frac{1}{g(y)} = \frac{1}{xg(y)}. \tag{18}$$

So a stringent test of the hypothesis is that $xg(y)$ be constant within any industry and over all countries for which we have data on capital. The stringency of the test comes from the fact that it relies on data (namely K) which have not been used in the previous analysis. If the test is passed, then not only have we validated the assumption of constant returns, but also (15) and with it our whole approach to the production function.

When $h(y)$ is obtained by solving for w in (8), the integration needed to determine $g(y)$ is a repetition of the argument leading to (10). Then,

$$\frac{1}{xg(y)} = \frac{\beta^{VP}}{K}(1 - \alpha y^{\rho})^{-1/\rho}.$$

Since β is a constant, a test of constancy of returns to scale is obtained by the condition that

$$c = \left(\frac{V}{K}\right)(1 - \alpha y^{\rho})^{-1/\rho} \tag{19}$$

is a constant.

2. *Capital and the rate of return.* It should not be overlooked that up to the previous paragraph our production functions have appeared only as rationalizations of the observed relation between y and w under assumptions about competition. We can not be sure that they do in fact describe production relations (i.e., holding among V, L, and K), and it is indeed intrinsically impossible to know this without data on K, or equivalently on the rate of return. Should such data be available, however, we can perform some further very strong tests of the whole approach.

Suppose we have observations on K for a particular industry across several countries. Then we know x as well as y and we can test directly whether our deduced production function (3) or (4) does in fact hold for some value of A. If it does, then this provides an estimate of A and a stringent external check on the validity of our approach.

This is merely a rephrasing of our test for constant returns, to emphasize that it really goes somewhat further; if the hypothesis of constant returns to scale is accepted, so is the validity of the implied production function.

3. *Neutral variations in efficiency.* From the argument leading to (10) and (11), it is seen that the parameters α and ρ are derived directly from our empirical estimates of a and b in section I. But β is a constant of integration and can be determined only from observed data including measurements of K or x. Now the test quantity c in (19) depends on α and ρ, but not on β, on the assumption that β is constant across countries. Failure of data to pass the stringent test based on (19) may be read as suggesting that β varies across countries while α and ρ are the same. From (11), this is equivalent to the statement that the efficiency of use

of capital varies from country to country, but not the efficiency of use of labor.

A more symmetrical (and more plausible) possibility is that international differences in efficiency affect both inputs equally. This amounts to assuming in (13) that the efficiency parameter γ varies from country to country while δ and ρ remain constant. Since $\beta/\alpha = \delta/(1-\delta)$, we can put this by saying that β and α vary proportionately. We can provide a test of this hypothesis.

From the definition of the elasticity of substitution and its constancy and the competitive equivalence of factor price ratios and marginal rates of substitution, it follows that w/r is proportional to $(K/L)^{1/\sigma} = (K/L)^{(1+\rho)}$. It is easy enough to calculate the constant of proportionality directly; we have

$$\frac{w}{r} = \frac{1-\delta}{\delta}\left(\frac{K}{L}\right)^{1+\rho} \tag{20}$$

and

$$\beta/\alpha = \delta/(1-\delta) = \left(\frac{r}{w}\right)\left(\frac{K}{L}\right)^{1+\rho} \tag{21}$$

Thus for countries from which we have data on r and K, and given our estimate of ρ for an industry, we may compare the values of the right-hand side of (21). If they are constant or nearly so, we conclude that there are neutral variations in efficiency from country to country, and we are able simultaneously to estimate δ. Then from δ and ρ we can use (12) to estimate the efficiency parameter γ in each country involved, for this particular industry.

4. *Factor intensity and the CES production function.* From (20) we see that:

$$x = \frac{K}{L} = \left(\frac{\delta}{1-\delta}\frac{w}{r}\right)^{\sigma}. \tag{20a}$$

Now imagine two industries each with a CES production function although with different parameters, and buying labor and capital in the same competitive market. Then

$$\begin{aligned}\frac{x_1}{x_2} &= \left(\frac{\delta_1}{1-\delta_1}\right)^{\sigma_1}\left(\frac{\delta_2}{1-\delta_2}\right)^{-\sigma_2}\left(\frac{w}{r}\right)^{\sigma_1-\sigma_2} \\ &= J\left(\frac{w}{r}\right)^{\sigma_1-\sigma_2}.\end{aligned} \tag{22}$$

If $\sigma_1 = \sigma_2$ (i.e., $\rho_1 = \rho_2$), then this relative factor-intensity ratio is independent of the factor price ratio. That is, industry one, say, is more capital-intensive than industry two, at all possible price ratios. This is the case both for the Cobb-Douglas function ($\sigma_1 = \sigma_2 = 1$) and the fixed-proportions case ($\sigma_1 = \sigma_2 = 0$). But once $\sigma_1 \neq \sigma_2$, this factor-intensity property disappears and it is impossible to characterize one industry as more capital-intensive than the other independently of factor prices. For (22) says quite clearly that there is always a critical value of w/r at which the factor-intensity ratio x_1/x_2 flips over from being greater than unity to being less. There is only one such critical value at which the industries change places with respect to relative capital-intensity. The nature of the switch is in accord with common sense: as wages increase relative to capital costs, ultimately the industry with the greater elasticity of substitution becomes more capital-intensive. Such switches in relative factor intensity should be observable if one compares countries with very different factor-price structures, which we have done for Japan and the United States in section IV.

The relative factor-intensity ratio plays an important role in discussions of the tendency of international trade in commodities to equalize factor prices in different countries (and for that matter, in the more general problem of the relation between factor prices and commodity prices in any general equilibrium system).

5. *Time series and technological change.* The CES production function is intrinsically difficult to fit directly to observations on output and inputs because of the non-linear way in which the parameter ρ enters. But, provided technical change is neutral or uniform, we may use the convenient factor-price properties of the function to analyze time series and to estimate the magnitude of technical progress.

A uniform technical change is a shift in the production function leaving invariant the marginal rate of substitution at each K/L ratio. From (13) and (20), uniform technical progress affects only the efficiency parameter γ, and not the substitution or distribution parameters, ρ or σ.

One notes from (20) that

$$\frac{wL}{rK} = \frac{1-\delta}{\delta}\left(\frac{K}{L}\right)^{\rho} \tag{23}$$

which is independent of γ. Hence if historical shifts in a CES function are neutral, (23) should hold over time, and its validity provides a test of the hypothesis of neutrality.

Suppose we have observations on x and on w/r at two points on the production function, say two countries or two points of time in the same country. Then, from (20a),

$$\frac{x_1}{x_2} = \left[\frac{(w/r)_1}{(w/r)_2}\right]^\sigma. \tag{24}$$

Thus an estimate of σ may be made. Note further that, since γ does not enter into (20a), the estimate is valid even if the efficiency parameter has changed between the two observations, provided the distribution and substitution parameters have not, that is, provided that technological change is neutral.[12]

If the hypothesis of neutrality is acceptable, we may try to trace the shifts in the efficiency parameter over time. One way to do this is to go back to (8). From $V/L = aw^b$ and the definitions of the parameters α, σ, δ, and γ, one calculates first that

$$\begin{aligned}\frac{wL}{V} &= \left(\frac{1}{a}\right)w^{1-b} = \alpha^\sigma w^{1-\sigma} \\ &= (1-\delta)^\sigma \gamma^{\sigma-1} w^{1-\sigma}.\end{aligned} \tag{25}$$

Two possibilities now present themselves. For given values of the parameters σ and δ, one can use (25) to compute the implied time-path of γ. Or alternatively one may assume a constant geometric rate of technological change, so that $\gamma(t) = \gamma_0 10^{\lambda t}$, and fit

$$\begin{aligned}\log\left(\frac{wL}{V}\right) &= [\sigma \log (1-\delta) + (\sigma-1) \log \gamma_0] \\ &\quad + (1-\sigma) \log w + \lambda(\sigma-1)t\end{aligned} \tag{26}$$

to estimate σ and λ. We return to this subject in section V.

6. *Variation in commodity prices among countries.* The accepted explanation of the variation in commodity prices among countries is based on differences in capital intensity and factor costs. In our production function the capital intensity depends both on σ and δ (instead of only on δ, as in the Cobb-Douglas function), and we also allow for differences in efficiency among sectors. The corresponding explanation of price differences is therefore more complex.

The price of a commodity in our model is defined as the direct labor and capital cost per unit of value added:

$$P = Wl + Rk = Wl\left(1 + \frac{R}{W} \cdot x\right),$$

where W and R are wages and return on capital in money terms (rather than using output as *numéraire*), $l = L/V = 1/y$, and $k = K/V$.

Substituting from (12) for the labor coefficient gives:

$$P = \frac{W}{\gamma} [\delta x^{-\rho} + (1-\delta)]^{1/\rho} [\frac{r}{W} \cdot x + 1] \tag{27}$$

in which the price of a commodity depends on factor costs and capital intensity. For a given production function, the ratio of prices in countries A and B can be stated as a function of the factor prices only by using (20a) to eliminate x:

$$\frac{P_A}{P_B} = \frac{W_A}{W_B} \cdot \frac{\gamma_B}{\gamma_A} \left[\frac{\left(\frac{\delta}{1-\delta}\right)^\sigma \left(\frac{w_A}{r_A}\right)^{\sigma-1} + 1}{\left(\frac{\delta}{1-\delta}\right)^\sigma \left(\frac{w_B}{r_B}\right)^{\sigma-1} + 1}\right]^{\frac{1}{1-\sigma}}. \tag{28}$$

The empirical significance of this result is discussed in section IV-C.

III. Tests of the CES Production Function

The CES function may describe production relations in an industry with varying degrees of uniformity across countries. Two tests were outlined in section II-D that enable us to make a tentative choice among three hypotheses: (i) all three parameters the same in all countries, (ii) same σ and one other parameter the same, (iii) only σ the same. The evidence presented in section A below rejects the first hypothesis but supports the second. Furthermore, there appears to be some uniformity in the efficiency levels of different industries in the same country; this possibility is analyzed in section B.

[12] This method of estimating the elasticity of substitution has been used by Kravis [9], 940–41.

In section C we investigate the possible sources of bias in our previous estimates of σ in the light of these findings.

A. *Generality of the Production Function*

The two tests given in (19) and (21) require estimates of either the capital stock or the rate of return on capital. Although such data are notoriously scarce and unreliable, we have been able to assemble comparable information on rates of return in four of the industries in Table 2 covering from three to five countries in each industry.[13] The capital stock can be estimated from the rate of return, r, by the relation: $K = (V-wL)/r$.

The test of constancy of all parameters involves the computation of c in equation (19). If there is no variation in efficiency, this number should remain constant. This calculation is shown in Table 3-A. The extent of variation in c is indicated by taking its ratio to the geometric mean for each industry, $\bar{c}$.

It is clear from these results that the hypothesis of constancy in all three parameters must be rejected, since there is a large variation in c in all four industries. We therefore abandon the idea that efficiency is the same among countries and look for constancy in either α, β, or δ. The first would imply that variations in efficiency apply entirely to capital (assumed in the computation of c in Table 3-A); the second that they apply entirely to labor; and the last that they affect both factors equally. The logic of the test was indicated in section II-D. The three possibilities are evaluated in Tables 3-B

TABLE 3. — TESTS OF THE CES PRODUCTION FUNCTION[a]

A. Test of Constant Efficiency[b]

	231. Spinning & Weaving			311. Basic Chemicals			341. Iron and Steel			350. Metal Products		
	w	r	$\frac{c}{\bar{c}}$	w	r	$\frac{c}{\bar{c}}$	w	r	$\frac{c}{\bar{c}}$	w	r	$\frac{c}{\bar{c}}$
United States	2920	.0526	1.72	4754	.2083	2.31	4387	.1984	2.96	4314	.1359	2.35
Canada	2708	.0403	1.16	4036	.2115	2.01	3769	.1740	2.13	3507	.1071	1.37
United Kingdom	874	.2022	1.44	..	..	..	1224	.1513	0.62	..	..	..
Japan	287	.1902	0.55	563	.2373	.70	664	.1911	0.49	422	.2245	0.31
India	276	.1543	0.63	320	.2200	.51	450	.2686	0.53	..	..	..

B. Test of Constant δ[c]

	β_i/α_i	δ_i	γ_i	β_i/α_i	δ_i	γ_i	β_i/α_i	δ_i	γ_i	β_i/α_i	δ_i	γ_i
United States	1.155	.536	1.016	3.283	.767	1.021	1.564	.610	1.733	1.039	.509	1.736
Canada	1.352	.575	.771	3.289	.767	.839	1.765	.638	1.378	1.180	.541	1.292
United Kingdom	1.514	.602	.880	..	..	..	1.265	.559	.857	..	..	..
Japan	1.833	.647	.428	2.785	.734	.567	1.461	.584	.687	.968	.492	.619
India	..	..	..	2.967	.748	.441	1.495	.599	.657	..	..	..
Mean		.590			.754			.598			.514	
Coefficient of Variation[d]		5.85 %			1.72 %			3.55 %			3.50 %	

C. Test of Constancy of α and β

	α_i	β_i	α_i	β_i	α_i	β_i	α_i	β_i
United States	.462	.534	.233	.765	.343	.536	.462	.480
Canada	.452	.611	.242	.796	.336	.593	.448	.529
United Kingdom	.410	.621	..	..	.458	.579	..	..
Japan	.431	.790	.297	.826	.444	.648	.535	.518
India	..	..	.298	.883	.442	.661	..	..
Mean	.439	.639	.268	.818	.405	.603	.482	.509
Coefficient of Variation[d]	4.17%	11.85%	11.19%	8.04%	12.84%	6.77%	7.41%	3.79%

[a] Sources of data: See text.
[b] Computed from equation (19).
[c] Computed from equations (21) and (12), using country values of δ in computing γ.
[d] Defined as $\frac{\Sigma_i \mid X_i - \bar{X} \mid}{N \cdot \bar{X}}$ where X_i is the country value, $\bar{X}$ is the industry mean, and N is the number of observations for the industry.

[13] The rates of return on capital were estimated from balance sheets of different industries. Capital was measured by net fixed assets (including land) plus cash and working capital. All financial investments were excluded. Total returns to capital were taken to be equal to gross profit from operations (excluding other income) minus depreciation. For further details see [12].

and 3-C by computing the coefficient of variation for each parameter in each industry.

Of these three possibilities, the constancy of δ, implying neutral variations in efficiency, is much the closest approximation while there is little to choose between the other two. For the four industries taken together, the coefficient of variation in δ is only 3.6 per cent, while it is more than twice as large for the other two parameters. We therefore tentatively accept equation (12) or (13) as the basic form of the CES production function. Constant σ and δ characterize an industry in all countries, and differences in efficiency are assumed to be concentrated in γ_i.

The conclusion that observations on the same industry in different countries do not come from the same production function is so important that it should be tested in a way that does not depend on our particular choice of a production function. If in fact all countries fell on the same production function, homogeneous of degree one, then a high wage rate must arise from a high capital-labor ratio, which must, in turn, imply a low rate of return on capital. From Table 3 we see there is by and large an inverse correlation between wages and rates of return, but the variation in the latter seems much smaller than is consistent with the wide variations in wage rates. This impression can be confirmed quantitatively with the aid of formula (7).

It is there noted that, for points on the same production function, homogeneous of degree one, the rate of return is a function of the wage rate, with an elasticity which is negative and equal in magnitude to the ratio of labor's share to capital's. We proceed as follows. Let v be the smallest observed value of this ratio. Then the elasticity of r, the rate of return, with respect to the wage rate, w, cannot exceed $-v$, so that,

$$\frac{\log\left(\frac{r_1}{r_0}\right)}{\log\left(\frac{w_1}{w_0}\right)} \leqq -v,$$

where the subscripts refer to any two countries. If we choose the countries so that $w_0 > w_1$, multiply through by the negative quantity log (w_1/w_0), and take antilogarithms, we find $r_1 \geqq r_0(w_0/w_1)^v = \underline{r}_1$, say. Then, if the two countries were on the same production function, the rate of return in the low-wage country could not fall below the limit $\underline{r}_1$.

For each of the industries in Table 3, a comparison was made of the rates of return in the lowest-wage country, with the corresponding lower bound $\underline{r}_1$, computed from the highest-wage country (the United States in each case). The results of this computation follow:

Industry	231	311	341	350
r_1	.190	.220	.269	.225
$\underline{r}_1$	.244	.667	1.213	1.749

Thus in each case the actual rate of return in the lowest-wage country falls below the theoretical minimum consistent with the assumption of a uniform production function, and in most cases very far below. The results of section A are thus strongly confirmed.

B. *Effects of Varying Efficiency*

Since we have revised our interpretation of the empirical evidence on the elasticity of substitution, we can no longer take the regression coefficient b in section I as equal to σ. We now present a formula for determining σ from b when efficiency is known to vary with the wage rate and indicate the magnitude of the correction involved. We then examine the residuals from the regression equations for further evidence of varying efficiency or other sources of bias in estimation.

1. *Estimation of σ.* It is plausible to assume that in each industry the efficiency parameter γ varies among countries with the wage rate. Since the wage rate increases with both γ and x, a country with high γ is also likely to have been more efficient in the past and to have had high income and savings. Thus we expect x to be positively correlated with γ across countries and γ to increase with w. Assume for convenience that this variation takes the form:

$$\left(\frac{\gamma_A}{\gamma_B}\right) = \left(\frac{w_A}{w_B}\right)^e \tag{29}$$

where the subscripts refer to countries A and B. The effect of variation in efficiency on the output per unit of labor (y) can be shown from (25) to be:

$$\frac{y_A}{y_B} = \left(\frac{\gamma_A}{\gamma_B}\right)^{1-\sigma}\left(\frac{w_A}{w_B}\right)^{\sigma}. \tag{30}$$

Substituting from (29) for the efficiency ratio and taking logs we get a formula comparable to equation (8) from which our elasticity estimates were derived:

$$\log \frac{y_A}{y_B} = (\sigma + e - e\sigma) \log \left(\frac{w_A}{w_B} \right). \qquad (31)$$

Comparing (8) and (31), we see that the regression coefficient b is equal to $(\sigma + e - e\sigma)$, or

$$\sigma = \frac{b - e}{1 - e}. \qquad (32)$$

Therefore it is only when efficiency does not vary with w — i.e. when $e = 0$ — that b is equal to σ. For $e > 0$, σ must be still smaller than b, and therefore, *a fortiori*, less than 1 when $b < 1$.

To get a rough idea of the magnitude of the correction, we normalized the γ's in Table 3 so that the United States value equals one in each case and then fitted log γ to log w by least squares.[14] We obtain the following result from the combined sample of 14 observations:

$$\log \gamma = \underset{(.043)}{.323} \log w - .039 \qquad R^2 = .82.$$

The separate industries vary somewhat, but the number of observations in each is too small for reliable estimates. Another source of information on e is provided by the comparisons of Japan with the United States in section IV, which cover 10 manufacturing industries. Here the median value of (γ_J/γ_U) is about .35, corresponding to a value of e of about .5.

For values of b less than 1, equation (32) shows that variation of efficiency with the wage rate will reduce the estimate of σ. Taking $e = .3$, values of b of .9, .8, and .7 yield values of .86, .71, and .57. At the median value of $b = .87$ observed in Table 2, the corresponding σ is .81.

2. *Residual variation by country.* The extent of the deviation of observed values of value added per unit of labor in each country from the values predicted by the regression equations is shown in Table 4. Apart from errors of observation, there are three main causes of these differences between the predicted value ($\hat{y} = a + b \log \frac{W}{P}$) and the observed value ($y = V/L$):

(a) Variations in efficiency, which affect L only.

(b) Variations in commodity prices, which affect both V and W/P. The net effect depends on the magnitude of the difference $(1 - b)$.

(c) Variations in the exchange rate, which affect V but not W/P.

The following deviations in each factor are associated with positive and negative values of $(y - \hat{y})$:

	Positive Residuals	Negative Residuals
Efficiency	Relatively efficient	Relatively inefficient
Commodity price	High	Low
Exchange rate	Overvalued	Undervalued

TABLE 4. — RESIDUAL VARIATION IN (V/L) BY COUNTRY[a]

			Number of Industries		
	Average V (1)	Average[b] (%) (2)	Above +5% (3)	Between +5% and −5% (4)	Below −5% (5)
United States	3841	+5	10	11	3
Canada	3226	+5	9	11	3
New Zealand	1980	+4	7	7	8
Australia	1926	−12	1	3	20
Denmark	1455	−8	2	6	16
Norway	1393	−9	2	6	15
Puerto Rico	1182	+22	9	1	8
United Kingdom	1059	−11	1	4	19
Colombia	924	+14	16	2	6
Ireland	900	−18	0	2	12
Mexico	524	+32	19	3	1
Argentina	519	+10	12	5	7
Japan	476	+7	9	5	9
Salvador	445	+12	10	1	5
Brazil	436	+33	9	0	1
Southern Rhodesia	384	−18	0	2	4
Ceylon	261	+7	5	0	6
India	241	−23	0	2	16
Iraq	213	+1	1	0	1

[a] Residual $\Delta y = y - \hat{y}$ derived from Table 2.
[b] Arithmetic mean of $(\Delta y/y)$.

Although we cannot separate these causes in countries for which we do not have observations of relative prices, the observed residuals help in the interpretation of our previous results. Of the five countries analyzed in Table 3, the United States, Canada, and Japan have small average deviations, and hence little country bias is introduced into any conclusions based on them. The United Kingdom and India have predominantly negative residuals in V/L, probably due to undervalued exchange rates. Correction of Table 3-B to allow for these possible

[14] This procedure is not strictly correct, since γ is computed from an assumed value of σ which is subsequently to be corrected, but it is roughly valid since γ is insensitive to variations in σ.

biases does not significantly affect the estimates of relative efficiency, however.

It seems plausible to interpret the systematic country deviations as due mainly to differences in exchange rates or in the level of protection. The United States, Canada, and Latin America have predominantly positive residuals, probably due to overvalued exchange rates and (in Latin America) high levels of protection. Western Europe and India have predominantly negative deviations, probably because of relatively undervalued exchange rates. Variations in exchange rates and prices introduce a bias in estimation only if they are systematically related to the wage rate, which does not seem to be the case.

Although another comparative study [5] strongly suggests the importance of economies of scale, their effects are not apparent here in the residual variation in V/L. Larger plant size may account for part of the higher efficiency and positive deviations in the United States, but any such effect in other countries having large markets is concealed by the other sources of variation.

C. *Effects of Price Variation*

Of the three sources of bias discussed in the preceding section, the variation in commodity prices is probably the least important because it has a similar effect on both variables in the regression analysis. Since some data on relative prices among countries are available, however, it is desirable to test the magnitude of the error introduced by ignoring prices.

When commodity prices are known, the regression equation of (8) should be restated as follows, using the commodity price as the *numéraire* for both value added and wages:

$$\log\left(\frac{V}{P}\cdot\frac{1}{L}\right) = a + b\log\left(\frac{W}{P}\right). \qquad (8a)$$

If prices are uncorrelated with wages, their omission affects the standard error but not the magnitude of the regression coefficient b. If prices are correlated with wages, the correction in the estimate of σ would be given by an equation similar to (32).[15] For example, an inverse relation between wages and prices would raise the estimate of σ for values of b less than 1.

[15] If $(P_A/P_B) = (W_A/W_B)^f$, then $\sigma = (b-f)/(1+f)$ if b is estimated from (8).

To test the quantitative significance of this correction, we have been able to assemble data for only two industries, neither of which corresponds entirely either in coverage or time to the original data.[16] Estimating b alternatively from equations (8) and (8a) for the eleven countries available gives the following results:

	Eq. (8)	Eq. (8a)
Furniture (260)	.814	.780
	(.045)	(.104)
Knitting mill products (232)	.692	.755
	(.035)	(.039)

In neither case is there a significant difference between the two estimates. Although this test by itself is by no means conclusive, such other evidence as is available on relative prices does not suggest that there are many sectors in which the estimate of σ would be significantly affected by this correction.

IV. Factor Substitution and the Economic Structure

Variations in production functions among industries have a substantial effect on the structural features of economies at different levels of income. In the present section, we shall investigate the effects on factor proportions, commodity prices, and comparative advantage that stem from differences in the parameters of the CES production function.

[16] The sectors covered are both consumer goods, since we were unable to find comparable data on intermediate products for any substantial number of countries. The prices used for sector 232 apply to all clothing rather than to 232 only. The price indexes are as follows for the 11 countries:

	Price of Furniture	Price of Knitted Goods
United States	100.00	100.00
Canada	154.70	148.91
Australia	81.95	73.58
New Zealand	94.82	103.46
United Kingdom	66.46	60.30
Denmark	89.41	77.13
Norway	94.37	89.90
Argentina	223.00	139.01
Brazil	145.90	97.20
Colombia	182.60	225.70
Mexico	175.80	124.58

Data are taken from *Internationaler Vergleich der Preis für die Lebenshaltung*, Ergänzungsheft Nr. 4 Zu Reiche 9, *Einzelhandelspreise in Ausland*, Verlag W. Kohlhammer GMBH, Stuttgart und Mainz, Jahrgang, 1959. The original data are in deutschmark purchasing power equivalents, from which the implied prices indexes were derived by taking the United States as a base. The exchange rates used in converting the prices to dollars were the ones that were used in section I.

To carry out this analysis, it is necessary to have some indication of the values of the three parameters in sectors of the economy other than those examined in section I, and hence to have some direct observations on the use of capital. For this purpose, we shall determine the parameters in the production function from data on comparable sectors in Japan and the United States. Although these two-point estimates may have substantial errors in individual sectors, the over-all results of this second method of estimation support the principal results of our earlier analysis and lead to some more general conclusions.

A. *Production Functions from U.S.-Japanese Comparisons*

The United States and Japan were selected for this analysis because of the availability of data on factor use, factor prices, and commodity prices in a large number of sectors.[17] They also are convenient in having large differences in relative factor prices and factor proportions. The errors involved in estimating the elasticity of substitution are therefore less than they would be if there were less variation in the observed values. (For the data in section I, estimates based only on the United States and Japan differed by less than 10 per cent on the average from the regression estimates.)

The elasticity of substitution can be estimated from these data by means of equation (24):

$$\frac{x_J}{x_U} = \frac{(K/L)_J}{(K/L)_U} = \left\{ \frac{\dfrac{w_J}{r_J}}{\dfrac{w_U}{r_U}} \right\}^{\sigma}$$

where subscripts indicate the country. This method of estimation has the advantage of utilizing direct observations of capital as well as labor and of being independent of the varying value of the efficiency parameter γ.

The data for this calculation are taken from input-output studies in the two countries and are summarized in Table 5. The main conceptual difference from section III is in the definition of capital, which here includes only fixed capital. The labor cost in Japan makes allowance for the varying proportions of unpaid family workers in each sector. The variation in relative factor costs shown in column (4) is due entirely to differences in labor costs, since the relative cost of capital is assumed to be the same for all sectors.

The values of σ derived by this method vary considerably more than those derived from wage and labor inputs alone in section I. However, for the 12 manufacturing sectors in which both are available, there is a significant correlation of .55 between the two estimates.[18] The weighted median of σ for these sectors is .93 as compared with .87 by the earlier analysis. The median σ is also .93 for all manufacturing. The omission of working capital provides a plausible explanation of this difference, since the little evidence available indicates that stocks of materials and goods in process are generally as high in low-wage as in high-wage countries. The elasticity of substitution between working capital and labor is therefore probably much less than unity. This correction is particularly important in trade and in manufacturing sectors having small amounts of fixed capital.

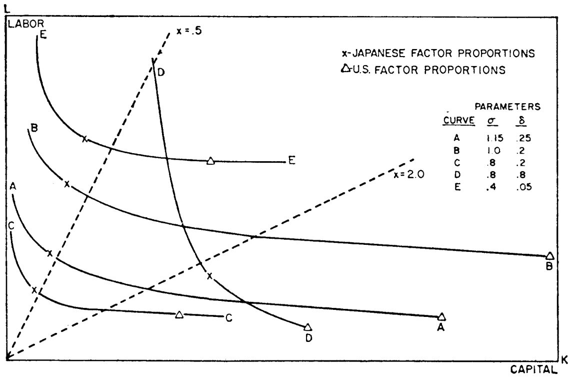
Since these two-country estimates are reasonably consistent with our earlier findings for the manufacturing sectors, we will tentatively accept them as indicative of elasticities of substitution in non-manufacturing sectors, with qualifications for the omission of working capital. Here the most notable results are the relatively high elasticities in agriculture and mining, and the low elasticity in electric power.[19] In trade, the omission of working capital probably leads to a serious overestimate of the elasticity of substitution, while for other services we have no comparable data. The evidence of relative prices, however, suggests an elasticity for personal services, at least, of substantially less than unity.

[17] The compilation of these data on a comparable basis has been done by Gary Bickel, who is conducting an extensive analysis of the relation between factor proportions and relative prices in the two countries. Further discussion of the data is given by Bickel [4].

[18] In some sectors the correspondence between the industries covered is very imperfect because the earlier estimates are on a 3-digit basis and cover only part of the 2-digit class.

[19] The transport sector involves a very large difference in product mix, and the reliability of the estimate is doubtful.

CHART 1. — C.E.S. PRODUCTIONS FUNCTIONS



On the basis of this comparison, we have constructed the five isoquants in Chart 1 to illustrate the range of variation in σ and δ. Sectors in Table 5 corresponding approximately to these sets of values are indicated in Table 6. The

TABLE 6. — ILLUSTRATIVE COMBINATIONS OF σ AND δ

	σ	δ	Examples
A	1.15	.25	Agriculture, mining, paper, non-ferrous metals
B	1.0	.2	Steel, rubber, transport equipment
C	.8	.2	Textiles, wood products, grain millling
D	.8	.8	Electric power
E	.4	.05	Apparel, personal services

effect of increasing σ in flattening the isoquant is shown by comparing E, C, B and A, while the effect of δ on the capital intensity is shown by comparing C and D. The optimum factor proportions at average Japanese and United States factor prices are also indicated to illustrate the discussion in the next section.

TABLE 5. — CALCULATION OF ELASTICITY OF SUBSTITUTION FROM FACTOR INPUTS AND FACTOR PRICES: JAPAN VS. UNITED STATES[a]

No.	Sector	Capital Intensity X_U (1)	X_J (2)	X_J/X_U (3)	Relative Factor Cost $(w_J/w_U \times r_U/r_J)$ (4)	Parameters Estimated σ (5)	δ (6)	Regression Estimate of σ (7)
	I *Primary Production*							
01, 02, 03	Agriculture	19.51	.367	.019	.036	1.20	.396	
04	Fishing	3.24	.490	.152	.133	.94	.201	
10	Coal mining	4.87	.534	.110	.093	.93	.182	
12	Metal mining	13.34	.566	.042	.107	1.41	.215	
13	Petroleum & natural gas	40.57	.722	.018	.096	1.71	.265	
14, 19	Non-metallic minerals	10.37	.777	.075	.111	1.18	.260	
	II *Manufacturing*							
205	Grain mill. production	5.36	.549	.103	.060	.81	.286	.91
20, 22	Processed food	5.11	.374	.073	.061	.93	.327	.82
23	Textiles	2.76	.340	.123	.073	.80	.159	.81
232, 243	Apparel	.99	.329	.332	.071	.42	.055	
241, 242, 29	Leather products	1.01	.190	.189	.098	.72	.051	.86
25, 26	Lumber and wood prod.	3.58	.310	.087	.054	.84	.198	.87
27	Paper	7.31	.528	.072	.099	1.14	.204	.96
28	Printing and publishing	3.45	.143	.042	.072	1.21	.092	.87
30	Rubber	3.73	.332	.089	.084	.98	.147	
31	Chemicals	8.32	1.125	.135	.157	.90	.325	.85
321, 329	Petroleum products	38.18	.360	.094	.151	1.04	.550	
322, 329	Coal	35.85	1.895	.053	.113	1.35	.365	
33	Non-metal. min. prod.	5.95	.414	.070	.084	1.08	.197	.95
341, 35	Iron and steel	8.60	.986	.115	.115	1.00	.273	.85
342	Non-ferrous metals	11.45	1.151	.101	.123	1.10	1.287	1.01
36, 37	Machinery	4.86	.469	.097	.083	.93	.187	.87
381	Shipbuilding	4.76	.477	.100	.094	.97	.174	
382	Transport equipment	5.01	.378	.075	.083	1.04	.169	
	III *Utilities and Services*							
511	Electric power	46.13	10.50	.228	.164	.82	.819	
61	Trade	5.93	.349	.059	.079	1.12	.187	
71	Transport	15.71	.316	.020	.106	1.74	.170	

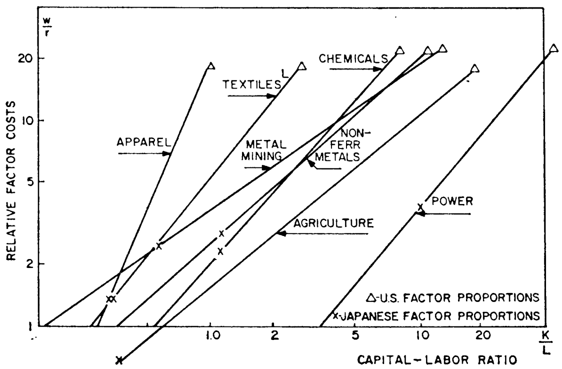
[a] SOURCES:
Cols. (1), (2), and (4) are taken from Bickel [4], based on U.S. and Japanese input-output materials; r_J/r_U assumed to be 1.47 for all sectors.
Col. (5) is calculated from equation (33).
Col. (6) is calculated from equation (20a).
Col. (7) aggregated from Table 2 using the average proportions of value added in the two countries as weights.

B. *Factor Costs and Factor Proportions*

As shown in equation (20a), the variation in factor proportions among countries, and among sectors in the same country, depends on σ, δ, and the relative factor costs. The U.S.-Japanese data are used in Chart 2 to provide a graphical illustration of both types of variation. When both variables are expressed as logarithms, the capital-labor ratio is a linear function of the relative factor cost:

CHART 2. — FACTOR COSTS AND OPTIMUM FACTOR PROPORTIONS

(Logarithmic Scale)



$$\log x = \sigma \log \left(\frac{\delta}{1-\delta}\right) + \sigma \log \frac{w}{r}.$$

In the United States, wages vary relatively little among sectors of the economy and the variation in capital intensity is due almost entirely to differences in δ and σ. In Japan, however, population pressure and underemployment are reflected in large wage differentials among sectors. Chart 2 shows that low wages rather than the production function account for the low capital intensity in Japanese agriculture; if there were as small a wage differential as in the United States, agriculture would become a relatively capital-intensive sector in Japan on this analysis. Similarly, high wages contribute to the relatively high capital intensity observed in Japan in sectors like power and non-ferrous metals.

As between countries, changes in the relative capital intensity of sectors have great significance for international trade. Our results indicate that changes in the ordering of sectors by factor proportions are normal rather than exceptional occurrences. The only sectors that are fairly immune to them are ones like power and apparel that have extremely high or low values of δ. A type that shifts its relative position a great deal is illustrated in Chart 2 by metal mining, which is quite capital intensive in the United States and quite labor intensive in Japan because of its high elasticity of substitution. For the less extreme cases, wage differentials of the magnitude of that between Japan and the United States will cause factor reversals even with relatively small differences in elasticity, but for smaller wage differences the ranking would be more constant.

Despite the approximate nature of our findings, the evidence of quantitatively significant reversals in capital intensity is too strong to be ignored.[20] The assumption of invariance in the ranking of commodities by factor intensity that is used by Samuelson [13] and other trade theorists appears to have very limited empirical application.

The varying possibilities for factor substitution also have important consequences for the allocation of labor and capital at different income levels. If there were no such variation, the distribution of labor and capital by sector would correspond to the distribution of output except for differences in efficiency.[21] In actuality, there are significant departures. Rising income leads to a declining share of primary production in total output and to an even more rapid decline in primary employment because of the high elasticity of substitution.[22] On the other hand, the observed rise in the share of labor in the service sectors as income rises is probably due primarily to a low elasticity of substitution, since the share of service output does not appear to rise significantly [5].

C. *Variations in Efficiency and Prices*

Although they have received most attention in trade theory, factor proportions are not the only determinants of relative prices. A complete explanation must also take account of dif-

[20] This aspect of our results and its implication for the problem of factor price equalization will be discussed by Minhas in a separate paper.

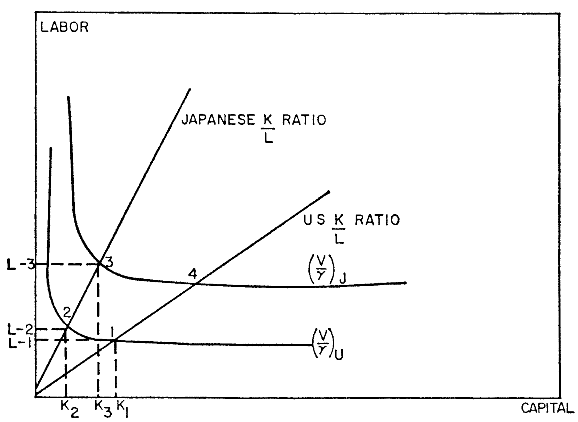
[21] Sector growth models using the Cobb-Douglas function are discussed by Johansen [8].

[22] In the case of underemployment, this tendency may be offset by low wages and low efficiency in agriculture, as in Japan.

ferences in relative efficiency among countries and industries, about which there is little systematic knowledge. We first present measures of relative efficiency in each sector derived from the Japan-United States comparison and then explore the combined effects of variation in all three parameters on relative prices among countries.

1. *Estimates of relative efficiency.* Having derived values of σ and δ in section IV-A from the properties of the production function alone, we can now estimate relative efficiency, γ_J/γ_U, by introducing information on relative commodity prices. The most direct method, which will be followed here, is to use commodity prices to determine isoquants for each country separately and then to derive the relative efficiency from a comparison of the isoquants. Other efficiency implications of observed prices are considered later.

CHART 3. — THE METHOD OF CALCULATING RELATIVE EFFICIENCY



This method of measuring relative efficiency between two countries is illustrated in Chart 3. We observe factor proportions and factor prices for the United States at point 1 and for Japan at point 3. The assumption of neutral efficiency differences enables us to derive σ and δ from these two observations, as was done in Table 5, and thus to determine the isoquants through each point. Given the labor input at point 1 (L_1), we can derive the labor input on the United States isoquant at point 2 from equation (12) according to the following formula:

$$\frac{V}{\gamma_U} = L_1\,[\delta x_U^{-\rho} + (1-\delta)]^{\frac{-1}{\rho}} = L_2\,[\delta x_J^{-\rho} + (1-\delta)]^{\frac{-1}{\rho}} \qquad (33)$$

where x_U and x_J are the capital intensities in the United States and Japan. Point 2 represents the combinations of factor inputs required to produce a unit of output with Japanese factor proportions and the United States efficiency level, γ_U. Point 3 represents the amount of labor and capital actually used in Japan to produce the same output, i.e., the labor and capital inputs per thousand dollars of output corrected for the differences in relative prices of the given commodity. Assuming neutrality in the effects on the two inputs, relative efficiency can be measured by the ratio of either labor or capital used at points 2 and 3:

$$\frac{\gamma_J}{\gamma_U} = \frac{L_2}{L_3} = \frac{K_2}{K_3}.$$

Given the properties of the CES production function, the measure of relative efficiency does not depend on the point chosen, and we obtain the same result by taking L_1/L_4. The calculation of relative efficiencies by this method is shown in Table 7 for all of the sectors from

TABLE 7. — CALCULATION OF RELATIVE EFFICIENCY: JAPAN VS. UNITED STATES[a]

		Values for Chart 3			Efficiency Ratio
ISIC No.	Sector	L_1	L_2	L_3	γ_J/γ_U
	Primary Production				
01–03	Agriculture	.083	.468	3.740	.13
10	Coal mining	.166	.245	1.724	.14
12	Metal mining	.117	.279	.876	.32
14, 15	Non-metallic min.	.143	.326	.583	.56
	Manufacturing				
20–22	Processed foods	.046	.083	.339	.25
23	Textiles	.104	.145	.332	.44
241, 242, 29	Leather products	.110	.123	.291	.42
27	Paper	.063	.112	.332	.32
31	Chemicals	.051	.093	.269	.35
321, 329	Petroleum prod.	.025	.095	.077	1.23
322, 329	Coal products	.047	.203	.215	.95
33	Non-metallic min. products	.122	.150	.859	.17
341, 35	Iron and steel	.083	.153	.171	.90
342	Non-ferrous met.	.058	.116	.169	.69

[a] SOURCES:
L_1 = U.S. labor input in man-years per $1000 of output from Bickel [4].
L_2 = Japanese labor input for the same output at U.S. efficiency level from equation (33).
L_3 = Actual Japanese labor input from Bickel [4].
$\gamma_J/\gamma_U = L_2/L_3$.

Table 5 for which relative commodity prices are also available.

In manufacturing, the median efficiency level is .43 (or .35 weighted by value added) which is about the same as the average ratio of Japanese and American efficiency determined in section III.[23] In primary production it is considerably lower, with agriculture and coal mining only one seventh of the American level.

Three factors may be suggested to explain these differences in relative efficiency.

(i) Limited natural resources doubtless explain a large part of the lower efficiency of capital and labor in primary production in Japan.

(ii) Competitive pressure in exports and import substitutes was suggested in [6] to be a cause of more rapid productivity increases in these sectors in Japan. Inefficient sectors (agriculture, mining, food, non-metallic mineral products) produce for the home market in Japan and are protected by either transport costs or tariffs from foreign competition.

(iii) Relatively efficient sectors in Japan (petroleum products, coal products, steel, nonferrous metals) are characterized by high capital intensity, large plants, and continuous processing. There may be technological reasons why it is easier to achieve comparable efficiency levels under these conditions.

Tests of these and other hypotheses regarding relative efficiency must await similar studies for other countries.

2. *Relative commodity prices.* Since we now have estimates of all three parameters in the production function, we can investigate their importance to the determination of relative commodity prices.

To do this, we substitute representative values of w/r[24] for the United States, Western Europe, and Japan in formula (28):

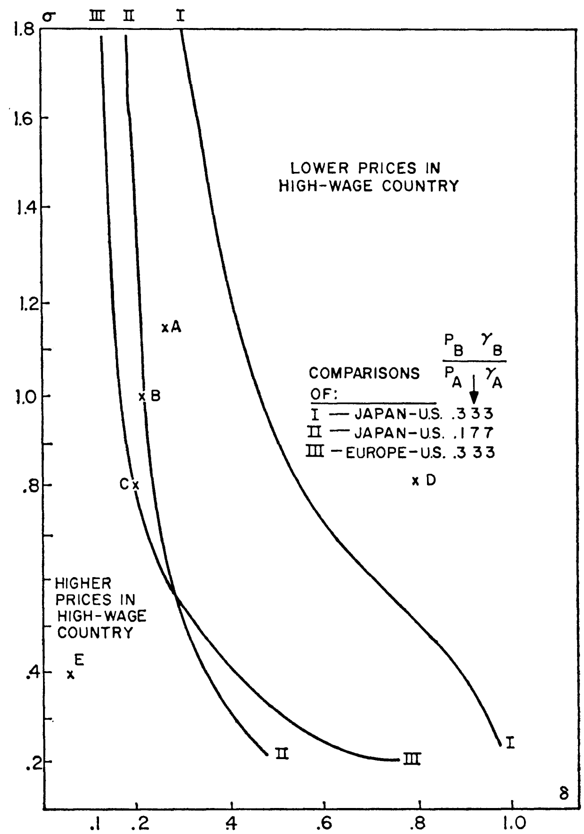
$$\frac{P_A}{P_B}=\frac{W_A\gamma_B}{W_B\gamma_A}\left[\frac{\left(\frac{\delta}{1-\delta}\right)^{\sigma}\left(\frac{w_A}{r_A}\right)^{\sigma-1}+1}{\left(\frac{\delta}{1-\delta}\right)^{\sigma}\left(\frac{w_B}{r_B}\right)^{\sigma-1}+1}\right]^{\frac{1}{1-\sigma}}$$

[23] None of the four examples in section III corresponds very well to the larger industries used here; the main difference in γ_J/γ_U is in steel, which shows a much lower efficiency in Table 3.

[24] Values of w/r assumed are: United States, 21.3; Europe, 5.0; Japan, 1.42.

and use it to construct curves of constant relative prices (isoprice curves). As indicated in section II-D-6, prices in our analysis refer only to the direct cost of labor and capital. Three such curves are shown in Chart 4. Curve I

CHART 4. — EFFECTS OF σ AND δ ON RELATIVE PRICES



assumes the typical efficiency ratio of .33 between Japan (B) and the United States (A), and equal prices. However, the average ratio of P_J/P_U actually calculated for the manufacturing sectors in Table 7 is about .53, a value which is illustrated by curve II. Since changes in efficiency have the same effect as changes in relative prices, however, curve II can also be interpreted as a line of equal prices and relative efficiency of .18. Curve III corresponds to curve I for the Europe-United States comparison, with an efficiency ratio of .33 and equal prices.

The general principle illustrated by Chart 4 is that high values of δ, σ, or the $\gamma_A\ \gamma_B$ ratio lead to lower prices in the high-wage country (A). A higher value of σ can offset a low value of δ to a considerable extent. Representative combinations of σ and δ are also shown in Chart 4, where the five illustrative sets of production parameters of Chart 1 are plotted. For a considerable number of the industrial sectors in Table 5, illustrated by the range of points A-B-C, variation in the elasticity of substitution seems to be more important than variation in δ in determining relative prices.

Actual price differences between Japan and the United States are affected as much by the cost of purchased inputs as by the value added component. Calculations for the ten manufacturing sectors in Table 7 based on (28) give a range of direct costs in Japan of .3 to .95 of the United States value, but this element is only about 35 per cent of total cost on the average. The average price of purchased inputs in these sectors ranges from .93 to 1.70 of their cost in the United States, which more than makes up for the lower cost of the factors used directly.[25] An adequate explanation of the differences in relative prices therefore requires an analysis of total factor use rather than of the direct use by itself.[26]

V. Substitution and Technological Change

A. *Technological Change, Labor's Share, and the Wage Rate*

1. *Historical changes in labor's share.* In section II-D-5, some implications of the CES production function for time series were derived. In particular (25), it was shown that labor's share was governed by the relation,

$$\frac{wL}{V} = (1-\delta)^{\sigma}\left(\frac{w}{\gamma}\right)^{1-\sigma}. \tag{25a}$$

Under the assumption of neutral technological change, the only parameter that varies is γ. For an elasticity of substitution less than one, labor's share rises when wage rates increase more rapidly than technological progress. (For $\sigma > 1$ the relation is reversed; for $\sigma = 1$, we are in the familiar Cobb-Douglas case where labor's share is independent of both neutral technological progress and wage-rate changes.) The historically observed relative constancy of labor's share can be understood in these terms; labor's share is the resultant of offsetting trends; and further, for σ not too far from 1, it is a relatively insensitive function of them.[27]

If in particular we add the assumption that technological change proceeds at a constant geometric rate, we have [cf. (26)]

$$\log\frac{wL}{V} = a_0 + a_1 \log w + a_2\, t, \tag{34}$$

where

$$a_0 = \sigma \log (1-\delta) + (\sigma-1) \log \gamma_0, \tag{35}$$

$$a_1 = 1-\sigma, a_2 = -\lambda(1-\sigma). \tag{35a}$$

From estimates of a_1 and a_2, it is easy, from (35a), to solve for estimates of λ and of the elasticity of substitution, which is $1 - a_1$. Equation (34) was fitted by least squares to the data for the United States non-farm production, 1909–49, given in Solow;[28] it was found that $a_1 = .431$ and $a_2 = -.003$. The corresponding estimate of σ is .569 and that of $\lambda = .008$, which corresponds to an annual rate of growth of productivity of 1.83 per cent. This figure agrees pretty well with most earlier estimates (Solow, [14], 316, gives 1.5 per cent; see also Abramovitz [1], 11).

We can test for the significance of the difference of the elasticity of substitution from its Cobb-Douglas hypothetical value of 1, which implies that a_1 and a_2 are both zero. The test then is equivalent to that for the significance of the multiple correlation coefficient, which has a value of .740; an F-test shows that, for 41

[25] Japan is somewhat exceptional among industrial countries in its dependence on imported materials, and between another pair of countries the variation in direct factor cost might be more indicative of the variation in total cost of production.

[26] The extent to which price differences between the two countries can be explained by total factor use is analyzed by Bickel [4]. He shows that the average capital intensity (reflecting δ) combined with the ratio of total factor intensities, which indicates a weighted average elasticity of substitution throughout the economy, gives a reasonably good prediction of relative prices. This result can be derived from equation (28) on the assumption that γ_J/γ_U does not vary greatly.

[27] However, (25a) does not provide a true causal analysis of the changes in labor's share if the wage rate is determined simultaneously with the other variables of the system. The wage rate may be treated as exogenous as in, for example, Lewis's model of economic growth [11], which is applicable to those economies in which there is large rural disguised unemployment.

[28] The data were derived from the columns of Table 1 in Solow [14] as follows: wL/V is obtained by subtracting column (4) (share of property in income) from 1; $w = (wL/V)(V/L)$, where V/L is column (5) (private non-farm GNP per man-hour); t is time measured in years from 1929.

observations, the value is highly significant. The alternative of capital-output and labor-output ratios which are fixed at any instant of time corresponds to $a_1 = 1$. Since the standard error of estimate of a_1 is .042, this hypothesis must also be rejected.

As in the cross-section studies, there is strong evidence that the elasticity of substitution is between zero and one. There may be some inconsistency with the results of the cross-section study of manufacturing sectors in section I in that the time series estimate of the elasticity of substitution is considerably lower than that found in the international comparisons. However, the time series data include services about whose elasticity of substitution we know little.

Estimation of (34) by least squares must be regarded only as an approximation. There is an unknown simultaneous equations bias in the procedure. However, more accurate methods would require a more detailed specification of the types of errors appearing in the marginal productivity relation (34) and the production function, a specification that we are not prepared to make. The results must therefore be regarded as tentative; the difficulties noted in section B below are undoubtedly related to the choice of estimation methods.

2. *A test of the production function.* As in the cross-section study, we can try to test the production function implied by the preceding results; this test, again, is a stringent one in that it uses capital data which have not been employed in fitting the marginal productivity relation. The fitting of (34) has yielded estimates of σ and λ but only one relation (35) between γ_0 and δ. We now use capital data to obtain separate estimates of these parameters. Define,

$$q = \frac{1}{\text{antilog}\left(\frac{a_0}{\sigma}\right)} \cdot$$

from (35), since

$$\rho = \left(\frac{1-\sigma}{\sigma}\right),$$

$$q = \frac{\gamma_0^{\rho}}{(1-\delta)} \cdot$$

If we let $\gamma = \gamma_0 (10)^{\lambda t}$ in the production function and express γ_0 in terms of q, we find, after some manipulation that,

$$X_1 - X_2 = \frac{\delta}{(1-\delta)} \qquad (36)$$

where,

$$X_1 = q\left(\frac{K}{V}\right)^{\rho} 10^{\lambda \rho t},\ X_2 = \left(\frac{K}{L}\right)^{\rho}. \qquad (36a)$$

As in the analysis of the international comparisons, the strictest test would be the constancy of the left-hand side of (36), which would imply an exact fit and simultaneously give an estimate of δ. In the absence of strict constancy, we can estimate $\delta/(1-\delta)$ as the average, $\bar{X}_1 - \bar{X}_2$; this yields the estimate, $\delta = .519$.[29] If we measure time in years from 1929, the value of a_0 was $-.080$, so that the value of γ_0 is .584. The production function for United States, non-farm output, 1929–49, is given by,

$$V = .584\,(1.0183)^t\,(.519\,K^{-.756} + .481\,L^{-.756})^{-1.322} \qquad (37)$$

We have tested this production function against actual output; the fit appears satisfactory. Out of 41 years, the prediction error is not more than 4 per cent of the predicted value in 22 years and not more than 8 per cent in 31. The maximum errors in prediction were -13.3 per cent (1933) and $+10.7$ per cent (1909).

It is further significant that all five of the years in which actual output fell short of predicted by more than 8 per cent were the depression years, 1930–34. This is reasonable on theoretical grounds; the immobility created by severe unemployment of both capital and labor causes inefficiency in the utilization of those resources that are employed (for a development of this argument, see Arrow [3]). If the depression years had been excluded we might have expected a still better fit.

B. *Relative Shares and the Capital-Labor Ratio*

Another test, with less satisfactory results, is that given by (23). In logarithmic form, the ratio of labor's to capital's share is related to the capital-labor ratio by:

[29] X_1 and X_2 are again calculated from Table 1 in [14]; K/L is given in column (6) (employed capital per man-hour); $K/V = (K/L)(V/L)$, where V/L is column 5.

$$\log\left(\frac{wL}{rK}\right) = \log\left(\frac{1-\delta}{\delta}\right) + \rho \log \frac{K}{L}. \qquad (38)$$

For ρ positive, the steady secular rise in the capital-labor ratio would give rise to an increase in labor's share, but one which would be very moderate unless ρ were large.

Relation (38) was fitted by least squares; the data were the same as those used in the preceding section.[30] The estimate of ρ was $-.095$, which implies an elasticity of substitution slightly greater than 1. The standard error of estimate of ρ is .098, so that the hypothesis of a positive value for ρ is not rejected, but the values are inconsistent with those found in section A.

The reasons for the contradiction of the two estimates are not clear. If (34) and the production function (37) both held exactly, then (38) would have to hold exactly with the same values of ρ and δ. Hence the discrepancy must be due to the different assumptions about the errors implicit in the statistical estimation methods. This problem remains an open one for the present.

Kravis ([9], 940–41) has applied essentially the same method, in the form of (24), to data for the entire economy (rather than only the non-farm portions, as here). His estimate of the elasticity of substitution is .64, which is much closer to the results of section A.

VI. Conclusion

This article has touched on a wide range of subjects: the pure theory of production, the functional distribution of income, technological progress, international differences in efficiency, the sources of comparative advantage. In part this broad scope reflects, as our introduction suggests, the fundamental economic significance of the degree of substitutability of capital and labor. In part it points to a wide variety of unsettled questions which are left for future research and better data. (In part, no doubt, it is simply due to the large number of authors of the paper!) Since our work does not lend itself to detailed summary, we content ourselves with a brief *reprise* of some of our findings, some speculation about others, and some suggestions for future research.

[30] $wL/rK = (wL/V)/[1 - (wL/V)]$.

A. *Findings*

We have produced some evidence that the elasticity of substitution between capital and labor in manufacturing may typically be less than unity. There are weaker indications that in primary production this conclusion is reversed. Although our original evidence comes from an analysis of the relationship between wages and value added per unit of labor, we have interpreted it by introducing a new class of production functions, more flexible and (we think) more realistic than the standard ones.

Although we began our empirical work on the naive hypothesis that observations within a given industry but for different countries at about the same time can be taken as coming from a common production function, we find subsequently that this hypothesis cannot be maintained. But we get reasonably good results when we replace it by the weaker, but still meaningful, assumption that international differences in efficiency are approximately neutral in their incidence on capital and labor. A closer analysis of international differences in efficiency leads us to suggest that this factor may have much to do with the pattern of comparative advantage in international trade.

Finally, our formulation contributes something to the much-discussed question of functional shares. If, on the average, elasticities of substitution are less than unity, the share of the rapidly-growing factor, capital, in national product should fall. This is what has actually occurred. But in the CES production function it is possible that increases in real wages be offset by neutral technological progress in their effect on relative shares.

B. *Speculation*

In his original work on what has since come to be known as the "Leontief scarce-factor paradox," Professor Leontief [10] advanced tentatively the hypothesis that the United States exports relatively labor-intensive goods not because labor is relatively abundant when measured conventionally, but because the efficiency of American labor is something like three times the efficiency of overseas labor. In our notation, this amounts to the suggestion that international differences in efficiency take the form of variations in β/α or, equivalently, of δ. We have proposed instead the hypothesis that β/α is constant across countries, while differences in efficiency are neutral. But we have also found some slight indications, in comparing Japan and the United States, that the American advantage in efficiency tends to be least in capital-intensive industries. This pattern, if it were verified, would seem to lead to an alternative interpretation of the Leontief phenomenon. But it also opens wide the question of why this association between differential efficiency and capital intensity should occur. Some possible explanations were mentioned in section IV-C, but the reader can think of others. We may be missing something important by excluding third factors, or external effects, or the importance of gross investment itself as a carrier of advanced technology into a sector.

Another active area of economic research where our results may have some interest is the theory of economic growth. An as yet unpublished paper [31] by J. D. Pitchford of Melbourne University considers the introduction of a CES production function into a macroeconomic model of economic growth and concludes that at least in some cases this amendment restores to the saving rate some influence on the ultimate rate of growth. Even more interesting are the possible implications for disaggregated general equilibrium models. Given systematic intersectoral differences in the elasticity of substitution and in income elasticities of demand, the possibility arises that the process of economic development itself might shift the over-all elasticity of substitution.

C. *Unsettled Questions*

Our general reference under this heading is *passim*. To begin with, as usable capital data for more countries and more industries become available, all of our results become subject to check for validity and generality. In particular, our speculations about the causes of varying efficiency are based primarily on comparisons between Japan and the United States. A more extensive study might easily controvert them.

Another loose end has to do with the question of returns to scale. We note that the stringent test for constant returns to scale and constancy of all parameters clearly has to be rejected. But it would be useful to explore the possibility of increasing returns to scale on a broader front. In view of equation (16) is there some choice of the function $C(K)$ which would yield a test on increasing returns? What light might this throw on the international comparisons, especially in connection with the less developed economies?

Finally, the whole question of further disaggregation calls out for exploration. We have in mind here not so much a finer industrial breakdown as a finer input breakdown. Can our labor and capital inputs be usefully subdivided? How about natural resource and purchased material inputs?

[31] Now published: "Growth and the Elasticity of Factor Substitution," *Economic Record*, December 1960, 491–500. (Note added in proof.)

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APPENDIX

International Data on Labor Inputs and Wage Rates

Industry	Dairy products		Canning and preserving of fruits and vegetables		Grain mill products		Bakery products		Sugar manufacturing and refining	
	202		203		205		206		207	
Country	L/V	$\$\omega$	L/V	$\$\omega$	L/V	$\$\omega$	L/V	$\$\omega$	L/V	$\$\omega$
1. United States (1954)	0.1256	3833	0.1449	2956	0.0903	3955	0.1472	3863	0.1203	3906
2. Canada (1954)	0.1860	2751	0.1859	2254	0.1254	3138	0.2224	2503	0.1002	3403
3. New Zealand (1955/56)	0.2003	2053	0.3410	1604	0.2311	1851	0.3679	1495		
4. Australia (1955/56)	0.2638	1886	0.3339	1715	0.2522	1916	0.3347	1680	0.2641	1846
5. Denmark (1954)	0.3758	1314	0.3735	1214	0.2562	1657	0.3600	1418	0.3348	1503
6. Norway (1954)	0.3170	1228	0.5472	1091	0.4932	1503	0.3073	1343		
7. United Kingdom (1951)	0.5077	972	0.5885	761	0.3775	1110	0.6467	846	0.3625	1195
8. Ireland (1953)	0.5019	910			0.4964	965	0.6469	857	0.4563	865
9. Puerto Rico (1952)	0.3180	1234	0.9270	484			0.5350	1015	0.3110	1637
10. Colombia (1953)	0.3480	937	0.3450	825	0.2090	653	0.6460	595	0.6710	854
11. Brazil (1949)										
12. Mexico (1951)	0.6188	495	0.7255	364	0.6821	340	0.8190	503	0.6341	524
13. Argentina (1950)	0.7437	396	0.6255	466	0.6507	585	1.7532	353	0.5145	681
14. El Salvador (1951)	0.5388	501			0.5040	495	0.5647	526	0.8525	178
15. Southern Rhodesia (1952)	0.7294	536			0.8475	402	1.2626	398		
16. Iraq (1954)										
17. Ceylon (1952)	0.5960	412	1.8150	236	2.0870	163				
18. Japan (1953)	0.5920	501	0.9472	391	0.6606	461	1.9185	253	0.2068	721
19. India (1953)			2.1500	165	5.1523	98	1.8200	236	2.4903	153

INTERNATIONAL DATA ON LABOR INPUTS AND WAGE RATES (*continued*)

Industry	Tobacco manufactures		Textiles (spinning and weaving)		Knitting mills		Lumber and wood		Furniture manufacturing	
	220		231		232		250		260	
Country	L/V	$\$\omega$	L/V	$\$\omega$	L/V	$\$\omega$	L/V	$\$\omega$	L/V	$\$\omega$
1. United States (1954)	0.0945	2738	0.2250	2920	0.2360	2698	0.2026	2993	0.1706	3515
2. Canada (1954)	0.1161	3023	0.2349	2708	0.2779	2260	0.2171	2546	0.2385	2668
3. New Zealand (1955/56)	0.2346	1562	0.3053	1755	0.3985	1548	0.2655	2002	0.3678	1834
4. Australia (1955/56)	0.2251	1810	0.3895	1559	0.3837	1487	0.3517	1794	0.3857	1713
5. Denmark (1954)	0.4044	1211	0.5024	1242	0.5004	1169	0.4905	1312	0.5040	1288
6. Norway (1954)	0.2888	1308	0.4993	1127	0.5046	1021	0.4714	1335	0.5228	1342
7. United Kingdom (1951)	0.2598	959	0.5560	874	0.6408	802	0.6359	964	0.6291	1078
8. Ireland (1953)	0.3500	1073	0.6813	708	0.8016	701	0.8288	850	0.8037	811
9. Puerto Rico (1952)			0.6290	932	0.7690	983	0.3340	1157	0.6240	861
10. Colombia (1953)	0.2960	826	0.3052	1089	0.5302	845	0.6310	696	0.7200	738
11. Brazil (1949)	0.5943	395	0.9334	343	0.9231	364	0.8611	347	0.9415	448
12. Mexico (1951)	0.1369	627	0.8673	461	0.7968	546	0.8584	358	0.9017	471
13. Argentina (1950)	0.2497	481	0.8212	481	0.8012	523	1.2050	367	1.0863	464
14. El Salvador (1951)	0.2470	909	0.8702	456	0.8032	542	1.1198	342	1.7825	377
15. Southern Rhodesia (1952)	1.1848	246					2.3201	298	1.2970	421
16. Iraq (1954)			2.1531	215						
17. Ceylon (1952)	1.0530	256	1.2190	226			2.0220	228		
18. Japan (1953)			1.3901	287	2.0526	252	1.8056	292	1.9668	323
19. India (1953)			2.5320	276			2.3840	188		

Industry	Pulp, paper and board		Printing and publishing		Leather tanning and finishing		Basic chemicals and fertilizers		Vegetable and animal oils and fats	
	271		280		291		311		312	
Country	L/V	$\$\omega$	L/V	$\$\omega$	L/V	$\$\omega$	L/V	$\$\omega$	L/V	$\$\omega$
1. United States (1954)	0.0945	4508	0.1254	4507	0.1645	3868	0.0814	4754	0.1037	3821
2. Canada (1954)	0.0924	4262	0.1736	3297	0.2104	2948	0.0973	4036	0.0793	3784
3. New Zealand (1955/56)	0.0998	2570	0.2655	2008	0.2851	2014	0.1603	2433	0.1576	2234
4. Australia (1955/56)	0.1927	2334	0.3118	1960	0.3283	2025	0.2341	2060	0.1250	2318
5. Denmark (1954)	0.2519	1428	0.3593	1683	0.3363	1702	0.2477	1553	0.2438	1698
6. Norway (1954)	0.2338	1545	0.5350	1457	0.4519	1391	0.2028	1612	0.2668	1524
7. United Kingdom (1951)	0.2503	1154	0.4838	1155	0.4822	1109	0.3641	1251	0.3559	1239
8. Ireland (1953)	0.6001	805	0.6221	1024	0.5746	903	0.4900	1178		
9. Puerto Rico (1952)	0.1130[a]	1426[a]	0.3190	1450	0.4510	989	0.2580	1647		
10. Colombia (1953)	0.6820	792	0.4590	1166	0.3070	1388	0.3050	1039	0.5840	591
11. Brazil (1949)	0.6355[a]	402[a]	0.5783	682	0.8800	328	0.4208[c]	481[c]		
12. Mexico (1951)	0.3831	676	0.7880	602	0.8787[b]	430[b]	0.3509	644	0.2167	710
13. Argentina (1950)	0.4016	644	0.7697	591	0.7836	572	0.5335	548	0.7146	561
14. El Salvador (1951)	0.9200	457	0.7502	624	0.7764	424	1.1740	298	0.7310	314
15. Southern Rhodesia (1952)										
16. Iraq (1954)										
17. Ceylon (1952)			1.3300	398	2.1890[b]	293[b]			0.7310	197
18. Japan (1953)	0.5077	647	0.8454	494	1.1112	439	0.5868	563	0.4460	507
19. India (1953)	1.3190	277			3.0100	185	0.9410	320	2.2422	157

[a] Refers to pulp, paper, board and paper products as well.
[b] Refers to leather tanning as well as leather products.
[c] Refers to all chemicals.

International Data on Labor Inputs and Wage Rates (*continued*)

Industry / Country	Miscellaneous chemicals 319 *L/V*	319 $ω	Structural clay products 331 *L/V*	331 $ω	Glass and glass products 332 *L/V*	332 $ω	Ceramics 333 *L/V*	333 $ω	Cement 334 *L/V*	334 $ω
1. United States (1954)	0.0720	4509	0.1712	3669	0.1228	4179	0.1958	3408	0.0752	4216
2. Canada (1954)	0.1120	3377	0.1606	3198	0.1926	3370			0.0635	3910
3. New Zealand (1955/56)	0.2189	1906	0.2126	2136	0.2457	2036	0.3172	1775	0.0686	2503
4. Australia (1955/56)	0.1871	1790	0.3351	2066	0.2856	1932	0.3828	1894	0.2562	2071
5. Denmark (1954)	0.2765	1575	0.4629	1329	0.4142	1416	0.4883	1320	0.2034	1738
6. Norway (1954)	0.2468	1473	0.5024	1358	0.3742	1484	0.5155	1187	0.2881	1569
7. United Kingdom (1951)	0.3600	1059	0.5941	1109	0.5477	1068	0.7666	835	0.2394	1431
8. Ireland (1953)	0.4745	943								
9. Puerto Rico (1952)	0.1680	1554	0.2800	800	0.5680	1079	0.7260	1184		
10. Colombia (1953)	0.2740	1094	1.0630	516	0.3290	968	0.6740	783	0.2040	1227
11. Brazil (1949)										
12. Mexico (1951)	0.3668	748	1.7452	321	0.6707	493	1.2285	328	0.2709	630
13. Argentina (1950)	0.5193	570	1.5700	370	0.8646	570	1.0480	440	0.5443	581
14. El Salvador (1951)	0.6830	429								
15. Southern Rhodesia (1952)										
16. Iraq (1954)										
17. Ceylon (1952)	0.8340	293	1.9340	167						
18. Japan (1953)	0.5270	486	2.5310	274	0.9660	467	1.8700	331	0.1693	1025
19. India (1953)	1.0970	337			5.4400	146	3.0470	229	0.8890	319

Industry / Country	Primary iron and steel 341 *L/V*	341 $ω	Primary non-ferrous metals 342 *L/V*	342 $ω	Metal products (except machinery) 350 *L/V*	350 $ω	Electrical appliances and equipment 370 *L/V*	370 $ω
1. United States (1954)	0.1266	4387	0.1119	4540	0.1332	4314	0.1314	4119
2. Canada (1954)	0.1410	3769	0.0768	3955	0.1562	3507	0.1579	3536
3. New Zealand (1955/56)	0.2340[d]	2190[d]			0.2639	2190	0.3122	1856
4. Australia (1955/56)	0.2338	2306	0.2351	2194	0.3268	1966	0.3431	1898
5. Denmark (1954)	0.2173	1656	0.1775	1695	0.4006	1441	0.4272	1466
6. Norway (1954)	0.2918	1568	0.2684	1648	0.4142	1438	0.3266	1489
7. United Kingdom (1951)	0.4503	1224	0.3870	1208	0.5464	979	0.6226	1024
8. Ireland (1953)								
9. Puerto Rico (1952)					0.2800	1307	0.1370	1535
10. Colombia (1953)	0.4199	1168	0.4560	1526	0.5640	879	0.4270	972
11. Brazil (1949)							0.5350	572
12. Mexico (1951)	0.4728	594			0.5580	455	0.5537	643
13. Argentina (1950)	0.7920	591	0.5512	542	0.8747	463	0.6826	619
14. El Salvador (1951)					2.7322	243		
15. Southern Rhodesia (1952)								
16. Iraq (1954)	1.2866	210						
17. Ceylon (1952)								
18. Japan (1953)	0.7256	664	0.5432	621	1.2388	422	0.8054	517
19. India (1953)	0.9860	450	2.2000	262			1.7170	306

[d] Refers to primary iron and steel as well as primary non-ferrous metals.