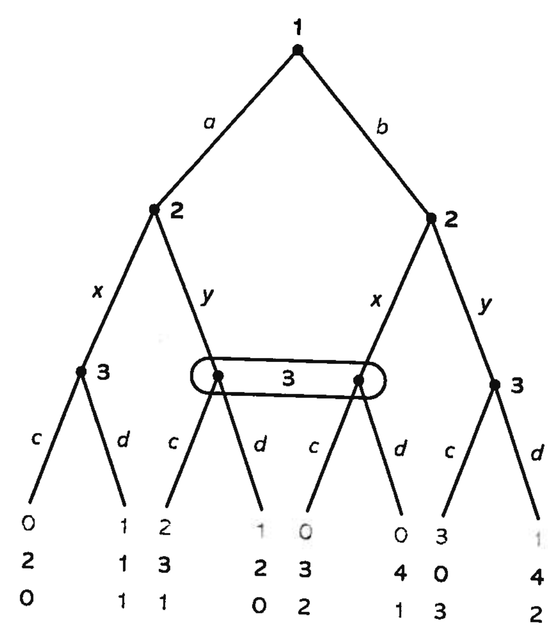
the substrategy profile form a Nash equilibrium for that subgame. Thus, SPNE mandates that players act optimally at every subgame, even those that are not reached if everyone were to act according to the SPNE. In this manner, when a player considers the ramifications of a particular move, she presumes that all players will respond in their best interests, not according to some idle threat.

Whereas backward induction requires that beliefs over *future* play should be consistent with rational behavior, **forward induction** requires that beliefs over *past* play should be consistent with rational behavior. Just as SPNE provides a method for selecting among Nash equilibria, forward induction provides a method for selecting among subgame perfect Nash equilibria. The approach is to eliminate those subgame perfect Nash equilibria for which the prescribed action at an information set is optimal only if the player holds beliefs about future play that are inconsistent with the past play's rationality.

In a game-theoretic context, **commitment** refers to a decision with some permanency that has an impact on future decision making. It may mean making an investment that influences the payoffs from various future actions or constraining the actions that can be chosen in the future. Commitment can be valuable in a strategic setting because of how it affects what other players do. A publicly observed commitment can bind a player to act a certain way in the future and can thereby induce other players to adjust their behavior in a way that may be desirable from the perspective of the player making the commitment. Understanding how commitment functions is a valuable piece of insight that game theory delivers.

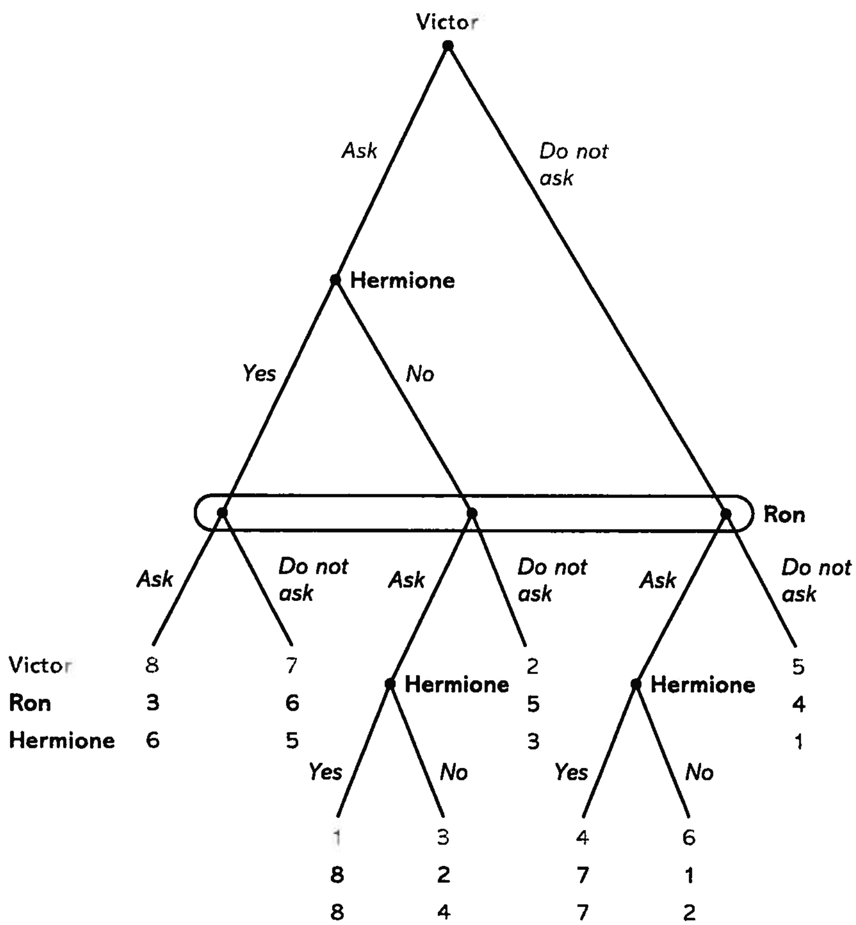
EXERCISES

1. Derive all subgame perfect Nash equilibria for the game below.



2. *"I can't come with you," said Hermione, now blushing, "because I'm already going with someone." "No, you're not!" said Ron. "You just said that to get rid of Neville." "Oh, did I?" said Hermione, and her eyes flashed dangerously. "Just because it's taken you three years to notice, Ron, doesn't mean no one else has spotted I'm a girl".*[4]

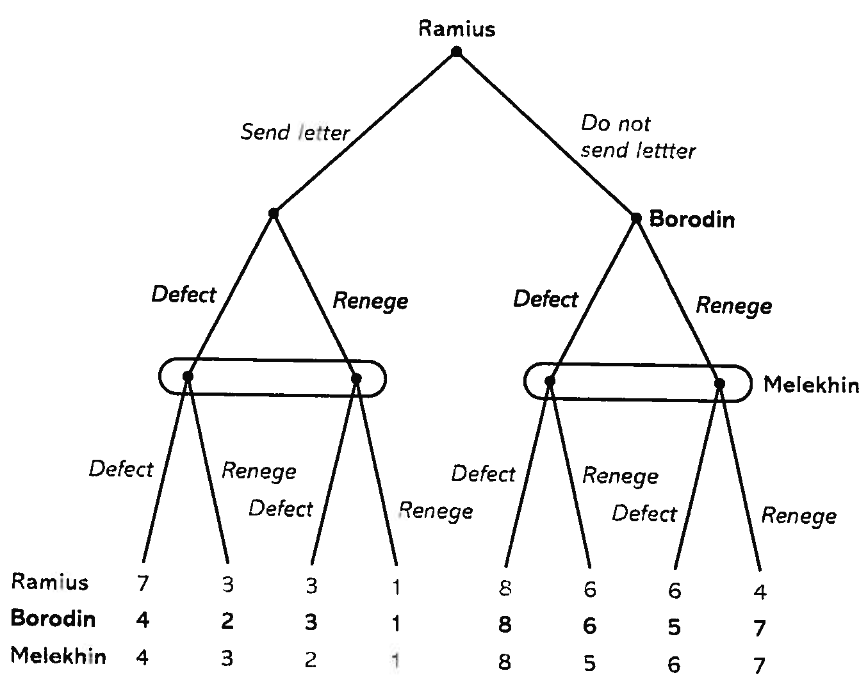
Yule Ball Dance



It is the week before the Yule Ball Dance, and Victor and Ron are each contemplating whether to ask Hermione. As portrayed above, Victor moves first by deciding whether or not to approach Hermione. (Keep in mind that asking a girl to a dance is more frightening than a rogue bludger). If he gets up the gumption to invite her, then Hermione decides whether or not to accept the invitation and go with Victor. After Victor (and possibly Hermione) have acted, Ron decides whether to conquer his case of nerves (perhaps Harry can trick him by making him think he's drunk Felix Felicis) and finally tell Hermione how he feels about her (and also invite her to the dance). However, note that his information set is such that he doesn't know what has happened between Victor and Hermione. Ron doesn't know whether Victor asked Hermione and, if Victor did, whether Hermione accepted. If Ron does invite Hermione and she is not going with Victor—either because Victor didn't ask, or he did and she declined—then Hermione has to decide whether to accept Ron's invitation. At those decision nodes for Hermione, she knows where she is in the game, since she is fully informed about what has transpired. The payoffs are specified so that Hermione would prefer to go to the dance with Ron instead of with Victor. Both Ron and Victor would like to go with Hermione, but both would rather not ask if she is unable or unwilling to accept. Use the concept of SPNE to find out what will happen.

3. For Exercise 2 on the Yule Ball Dance, now assume that, before he decides whether to ask Hermione, Ron observes whether or not Victor asked her. However, if Victor does invite Hermione, Ron does not know her answer to Victor when he decides whether to invite her himself.
   a. Write down the extensive form game.
   b. Derive all subgame perfect Nash equilibria.
4. In Tom Clancy's novel *Hunt for Red October*, the Soviet Union has developed a submarine named *Red October* that can run "silently" and thereby escape detection. On its maiden voyage, the ship's captain, Marko Ramius, has decided to defect, because he believes that this technology risks war by destroying the balance of power between the United States and the U.S.S.R. He has put together a set of officers who are loyal to him and have agreed to defect as well. However, the captain is concerned that an officer may change his mind during the voyage and, furthermore, that an officer may

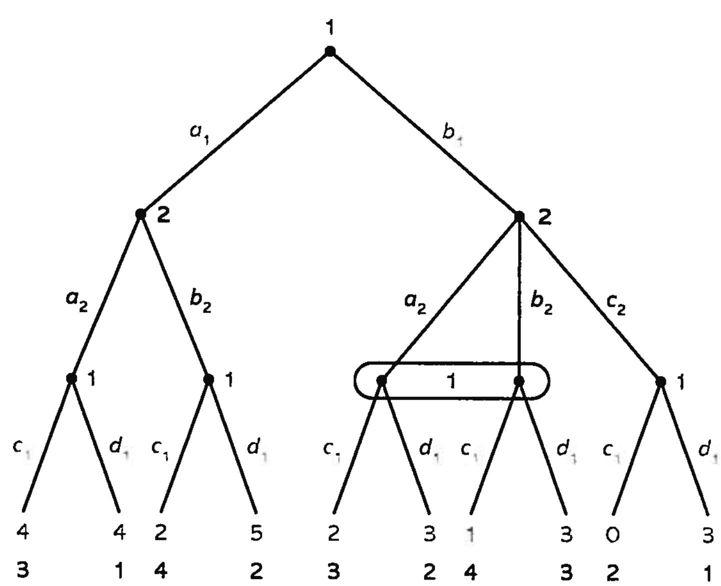
Hunt for Red October



be more inclined to change his mind if he thinks that other officers will do so. The captain is then considering writing a letter—to be delivered to the Soviet government after the submarine has departed from its base—stating his plan to defect. The extensive form of this game is shown on page 346. The captain initially decides whether or not to send the letter. After revealing his decision to his officers (once they are all out to sea), the officers, which, for the sake of parsimony, are limited to Captain Ramius, Second Rank Borodin and Lieutenant Melekhin, simultaneously decide between continuing with the plan to defect or reneging on the plan and insisting that the submarine return to the Soviet Union. The payoffs are such that all three players would like to defect and would prefer that it be done without the letter being sent (which results in the Soviet government sending out another submarine to sink *Red October*).

a. Derive all subgame perfect Nash equilibria.
b. Explain why the captain would send the letter.

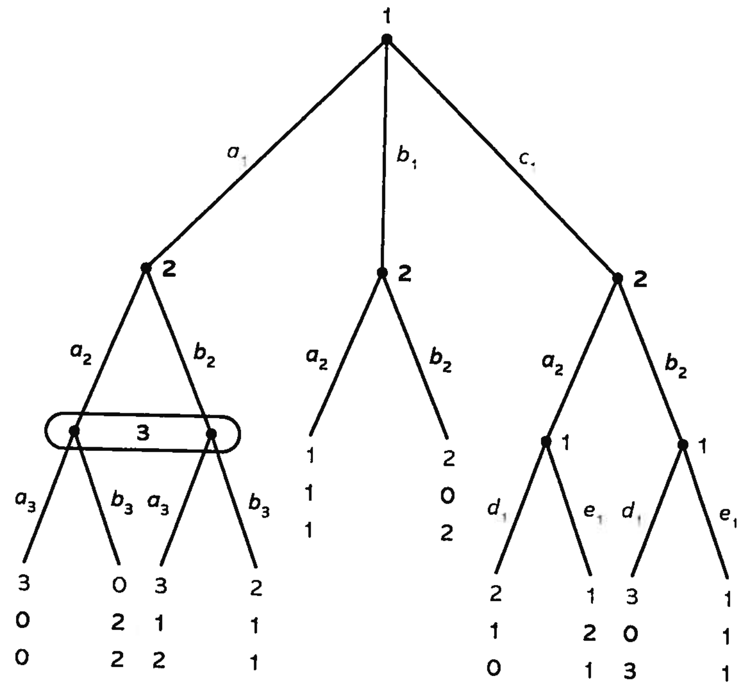
5. Consider the extensive form game shown here. The top payoff at a terminal node is for player 1. Find all subgame perfect Nash equilibria for the game below.



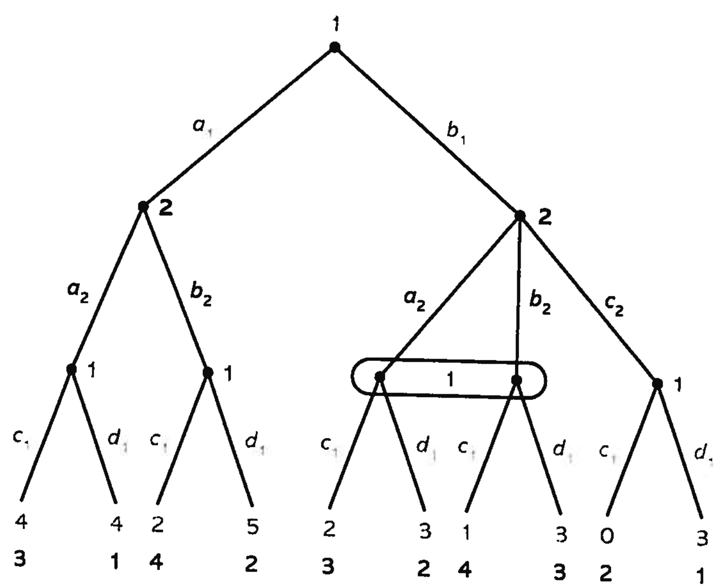
6. Consider the extensive form game portrayed below. The top number at a terminal node is player 1's payoff, the middle number is player 2's payoff, and the bottom number is player 3's payoff.

a. Derive the strategy set for each player. (*Note:* If you do not want to list all of the strategies, you can provide a general description of a player's strategy, give an example, and state how many strategies are in the strategy set.)
b. Derive all subgame perfect Nash equilibria.

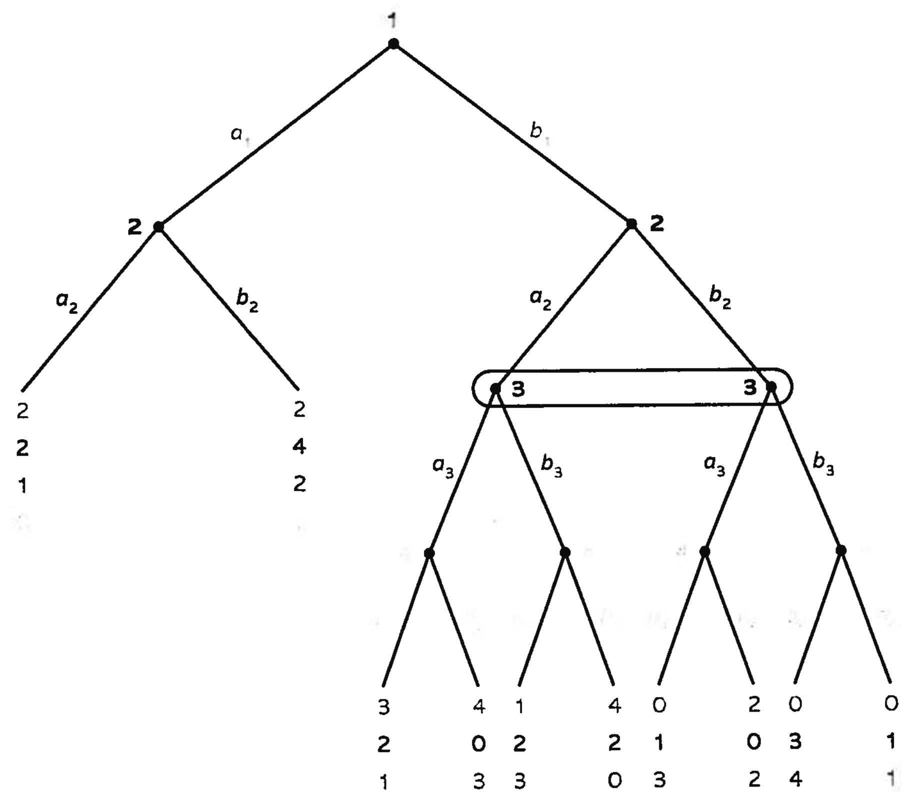
c. Derive a Nash equilibrium that is not a SPNE, and explain why it is not a SPNE.



7. Consider the extensive form game shown below. The top number at a terminal node is player 1's payoff and the bottom number is player 2's payoff.
   a. Describe the general form of each player's strategy.
   b. Derive all subgame perfect Nash equilibria.



8. Consider the extensive form game below. The top number at a terminal node is player 1's payoff, the second number is player 2's payoff, the third number is player 3's payoff, and the bottom number is player 4's payoff.
   a. Derive the strategy set for each player or, alternatively, state a representative strategy for a player.
   b. Derive all subgame perfect Nash equilibria.

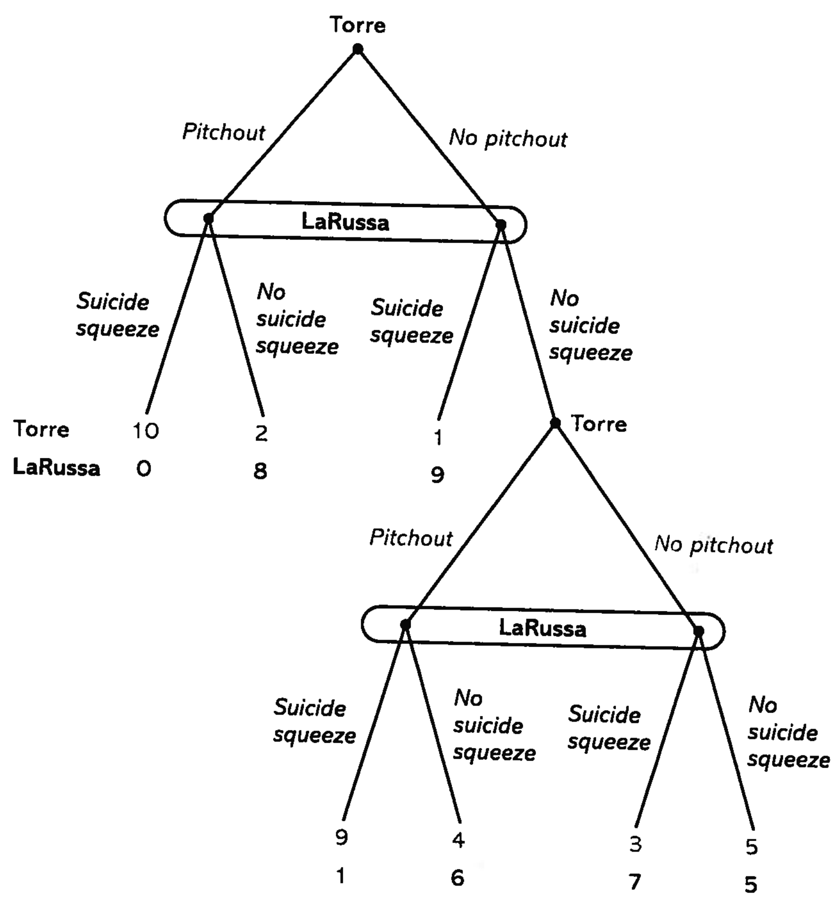


9. One of the most exciting plays in baseball is the "suicide squeeze." The situation involves a runner on third base and fewer than two outs. As soon as the pitcher is in his windup and committed to throwing the ball to home plate, the runner makes a mad dash for home plate. The batter's task is to square up and bunt the ball away from home plate so that no one has the chance to field the ball and tag the runner out. The other team can obstruct this play by performing a "pitchout": the pitcher intentionally throws a pitch so wide off the plate that the batter is incapable of getting his bat on the ball. The catcher, knowing that the pitchout is coming, steps over to catch the pitch and easily tags the runner out coming from third base. Of course, the manager for the team at bat may sense that a pitchout is planned and call off the suicide squeeze. If the other manager does call for

a pitchout, but no suicide squeeze occurs, he has put the pitcher in a more difficult situation, because the batter's count of balls and strikes will now be more to the batter's favor.

Let's consider such a situation as faced by two excellent and strategic-minded managers: Tony LaRussa (who is an avid believer in the play) and Joe Torre. The situation is as depicted in the accompanying figure. To simplify matters, we'll assume that the current count on the batter is, say, two balls and one strike, so that Torre, whose team is pitching, can (realistically) call at most one pitchout. Initially, the two managers move simultaneously, with Torre deciding whether to call for a pitchout and LaRussa deciding whether to execute a suicide squeeze. If a pitchout and a suicide squeeze both occur, the outcome will be disastrous for LaRussa and spectacular for Torre; the latter gets a payoff of 10, the former 0. If there is a pitchout and no suicide squeeze, then LaRussa's payoff is 8 and Torre's is 2. If there is a suicide squeeze and no pitchout, then the outcome is exactly as LaRussa wants, and his payoff is 9, with Torre receiving 1. Finally, if neither a pitchout nor a suicide squeeze occur, then the strategic situation is presumed to continue with the next

The Suicide Squeeze Game



pitch, when again Torre can call for a pitchout and LaRussa can execute a suicide squeeze. Find a SPNE in mixed strategies.

10. On a street in Boston, two food vendors—both selling Philly cheese steaks—are deciding where to locate their carts. A vendor can locate her cart on one end of the block (denoted location 0), in the middle of the block (denoted location 1/2), or on the other end of the block (denoted location 1). The two vendors simultaneously decide where to position their carts on the street. Let v_i denote the location of vendor i. Therefore, nine different location outcomes (v_1,v_2) can occur: (0,0),(0,1/2),(0,1),(1/2,0),(1/2,1/2),(1/2, 1),(1,0),(1,1/2),(1,1). Note that they are allowed to locate at the same spot on the street. After having located their carts—and knowing where both carts are located—they simultaneously choose a price for their cheese

$(v_1, v_2) : \{(0, 0), (\frac{1}{2}, \frac{1}{2}), (1, 1)\}$

Vendor 2

Vendor 1		Low	Moderate	High
	Low	4,4	7,1	9,0
	Moderate	1,7	6,6	12,5
	High	0,9	5,12	10,10

$(v_1, v_2) \in \{(0, \frac{1}{2}), (1, \frac{1}{2})\}$

Vendor 2

Vendor 1		Low	Moderate	High
	Low	3,5	4,6	7,3
	Moderate	2,7	5,8	8,6
	High	1,9	3,11	7,13

$(v_1, v_2) \in \{(\frac{1}{2}, 0), (\frac{1}{2}, 1)\}$

Vendor 2

Vendor 1		Low	Moderate	High
	Low	5,3	6,4	3,7
	Moderate	7,2	8,5	6,8
	High	9,1	11,3	13,7

$(v_1, v_2) \in \{(0, 1), (1, 0)\}$

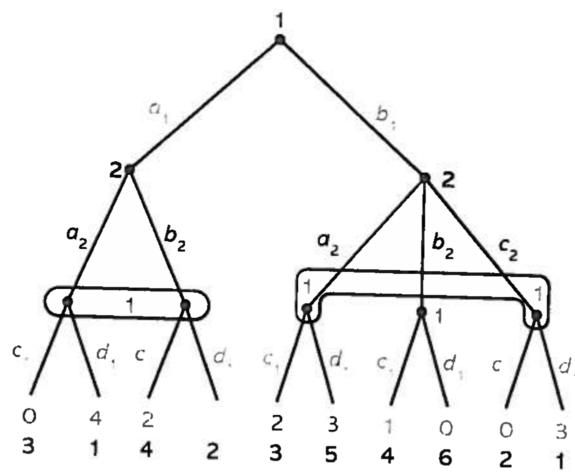
Vendor 2

Vendor 1		Low	Moderate	High
	Low	4,4	5,5	7,4
	Moderate	5,5	6,6	8,7
	High	4,7	7,8	10,10

steaks. Three possible prices are used: *low, moderate,* and *high*. Depending on the locations selected, the figure provides the payoffs associated with any pair of prices.

a. For each pair of locations, find the Nash equilibrium prices that the vendors charge.
b. Using SPNE, where do the vendors locate and what prices do they charge?

11. The 1978 film *Norma Rae* is based on a true story about Crystal Lee Sutton who was a textile worker in Roanoke Rapids, North Carolina. Crystal was inspired to help workers at the J.P Stevens textile mill form a union. In the film and in real life, she stands on her worktable and holds up a sign that says "UNION" and stays there until all of the machines are silent. Though fired from her job, the mill did become unionized, and she later went to work as an organizer for the textile union. In modeling this situation, Norma Rae is assumed to decide whether or not to join the union. Upon observing Norma Rae's choice, the other $n \geq 1$ workers simultaneously choose whether or not to join the union. For all workers (including Norma Rae), the payoff is 2 from not joining, 1 from joining when the fraction of workers who join is less than two-thirds, and 3 from joining when the fraction of workers who join is at least two-thirds. Assume that a worker joins the union when she is indifferent about joining.*



a. Find the SPNE. (Hint: The answer depends on the value of n).
b. Now suppose that payoffs are as follows. For Norma Rae, her payoff from not joining is 0, from joining when at least 2 people join is 1, and

*Thanks to Grace Harrington for suggesting this exercise.

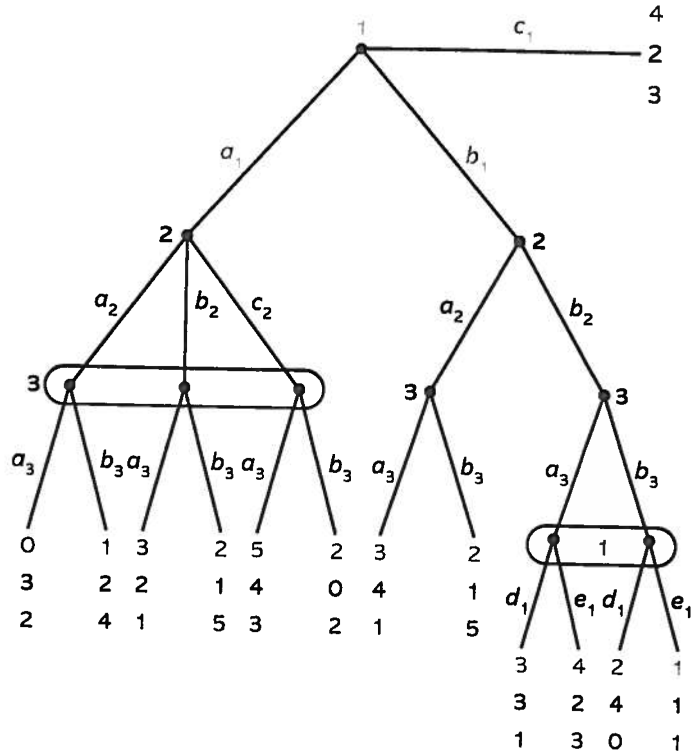
from joining when less than 2 people join is -1. For the other n workers, we will denote them as worker #1, worker #2, on through worker #n. For worker #i, his payoff from not joining is 0, from joining when at least $1 + i$ people join is 1, and from joining when less than $1 + i$ people join is -1, where $i = 1, \ldots, n$. Find the SPNE.

12. Consider the extensive form on the preceding page. The top number is player 1's payoff and the bottom number is player 2's payoff.
    a. Describe the general form of each player's strategy.
    b. Find the subgame perfect Nash equilibria.
    c. Find the Nash equilibria. (Hint: Write down the strategic form of the game.)

13. In the film *The Italian Job*, Steve is trying to move his stash of gold without his nemesis Charlie knowing where the gold is going. Steve knows that Charlie is watching Steve's house, where the gold is stored. Steve's plan is to have multiple armored trucks pull into his multi-car garage and to put the gold in one of the trucks. Charlie must then figure out which truck has the gold. Before all that happens, Steve decides whether to rent two or three trucks, where the cost of a third truck is 2 units of payoff. Charlie observes how many trucks arrive, so he knows how many trucks were rented. Steve decides whether to put the gold in truck A or B (when he rents two trucks) or in truck A, B, or C (when he rents three trucks). After Steve has loaded the gold onto one of the trucks, Charlie decides which truck to follow (A or B when two trucks are used; A, B, or C when three trucks are used) without knowing which truck has the gold. If Charlie ends up following the truck with the gold, his payoff is 6 and Steve's payoff is 0. If Charlie follows a truck without the gold, his payoff is 0 and Steve's payoff is 6. Recall that if Steve chose three trucks, then 2 must be subtracted from his payoff. (Note: In the film, Charlie had his computer geek Lyle figure out which truck had the gold by measuring how much the tires were compressed and thus which truck was carrying a heavier load. We're not allowing for that here.)
    a. Write down the extensive form of the game.
    b. For the subgame in which Steve chose two trucks, find the Nash equilibrium in mixed strategies.
    c. For the subgame in which Steve chose three trucks, find the Nash equilibrium in mixed strategies.
    d. For the game, find an SPNE in mixed strategies.

14. Consider this extensive form game on the top of page 354. The top number is player 1's payoff, the middle number is player 2's payoff, and the bottom number is player 3's payoff.
    a. Describe the general form of a strategy for each player.
    b. Find the SPNE.

15. Consider the game on the bottom of page 354. The top number is player 1's payoff and the bottom number is player 2's payoff.
    a. Find the SPNE.
    b. Find a Nash equilibrium that is not an SPNE.


1
c_1
4
2
3
a_1
b_1
2
2
a_2
b_2
c_2
a_2
b_2
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a_3
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1
0 3 2
1 2 4
3 2 1
2 1 5
5 4 3
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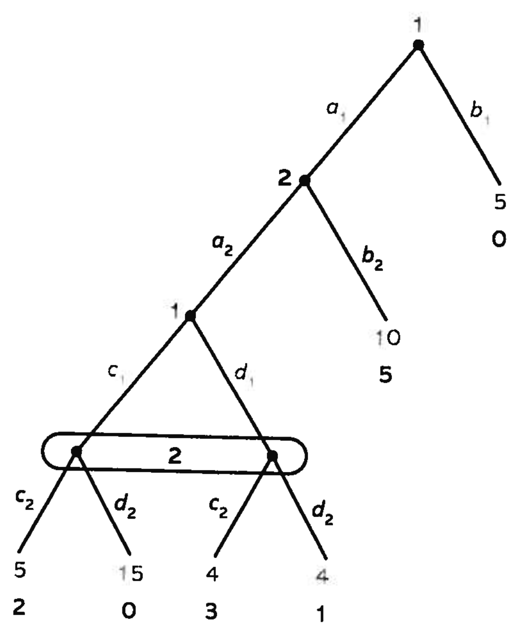
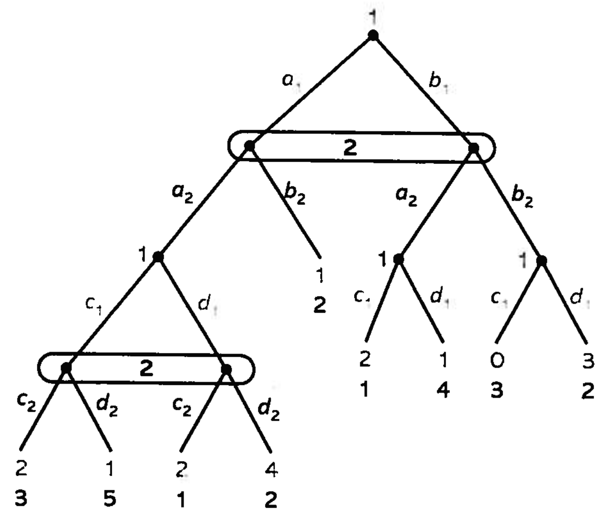
4 2 3
2 4 0
1 1 1


1
a_1
b_1
2
5 0
a_2
b_2
1
10 5
c_1
d_1
2
c_2
d_2
c_2
d_2
5 2
15 0
4 3
4 1

16. Consider the game below. The top number is player 1's payoff and the bottom number is player 2's payoff.
    a. Describe the general form of a strategy for each player.
    b. Find the SPNE

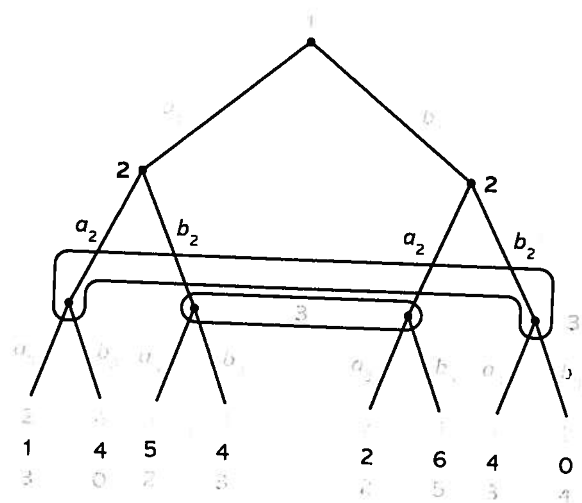


17. Player 1 is seeking to get a project approved by the local government, which will require bribing a few government officials. Three officials are relevant to the approval process: players 2, 3, and 4. Player 1 has access to player 2, while player 2 has access to players 3 and 4. In stage 1, player 1 offers a bribe to player 2, denoted b_2, which is any integer from {0,1, . . . , 100}. The bribe is of the form: "If the project is approved, then I'll pay you b_2." In stage 2, player 2 accepts or rejects the bribe. If he rejects it, then the project is not approved and the game is over. If player 2 accepts the bribe (which, recall, is paid only if the project is eventually approved), then the game moves to stage 3, which has player 2 simultaneously offer bribes to player 3 and 4, which are denoted b_3, and b_4, respectively. b_3, and b_4 are any integers from {0, 1, . . . , b_2,} for which their sum does not exceed b_2; thus, player 2 can offer bribes financed by the bribe given to him by player 1. In stage 4, players 3 and 4 each decide whether to accept or reject the bribe made to him by player 2. If both players 3 and 4 reject the bribes, then the project is not approved, no bribes are paid to any players, and the game is over. If, out of players 3 and 4, at least one player accepts a bribe, then the project is approved and all bribes are paid to those players who accepted bribes. Player 1's payoff is 0 when the project is not approved and is 100 minus the bribe paid to player 2 when the project is approved. For player 2, if the project is not approved, then his payoff is 4.5; if the project is approved, then it equals the bribe he received from player 1 less the bribes he paid to players 3 and 4. For player 3 (4), his payoff equals 9.5 (14.5) when the project is not approved and/or he rejected the bribe, and equals the bribe he was offered when the project is approved and he accepted the bribe. Using SPNE, is the project approved, and, if so, what bribes are given out?

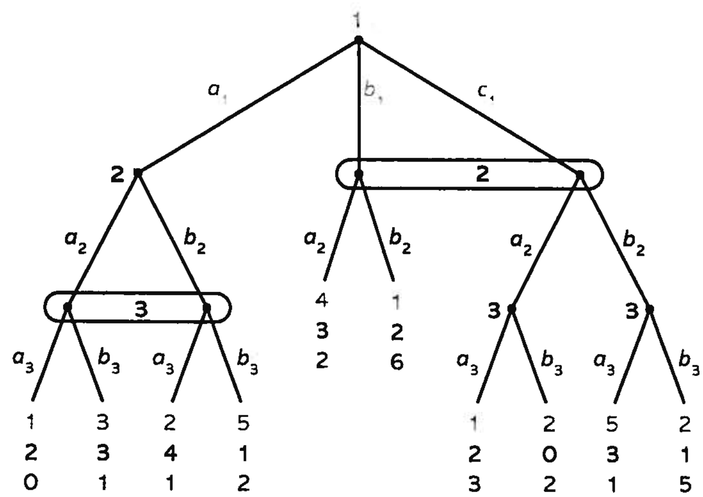
18. The owner of an item has decided to sell it using a first-price sealed bid auction. Three bidders are participating. Bidder 1 assigns a value of 10.7 to the item, bidder 2's value is 15.3, and bidder 3's value is 19.4. The game has two stages. In stage 1, the seller selects a reserve price r, which is the lowest price for which she'll sell it at auction. The reserve price is allowed to be any non-negative integer. In stage 2, the three bidders simultaneously submit bids, after having learned the reserve price. A bid is any non-negative integer that does not exceed a bidder's value. If the highest bid is greater than or equal to r then the item is awarded to the highest bidder who then pays a price equal to her bid. In that case, the winning bidder's payoff is her value minus the price paid for the item. If a bidder does not win the item, then her payoff is 0. If two bidders submitted the highest bid (and it is at least as great as the reserve price), then each has probability 1/2 of being declared the winner (and getting the item at a price equal to her bid) and probability 1/2 of not being the winner (with payoff of 0). If all three bidders submitted the highest bid (and it is at least as great as the reserve price) then each has probability 1/3 of being declared the winner (and getting the item at a price equal to her bid) and probability 2/3 of not being the winner (with payoff of 0). If the item is sold at auction, then the seller's payoff equals the price for which it is sold, and her payoff is 0 if it is not sold.

a. Describe the general form of a strategy for each player.
b. Using SPNE, what is the reserve price, who wins the item, and what is the winning bid?

19. Consider this extensive form game. The top number is player 1's payoff, the middle number is player 2's payoff, and the bottom number is player 3's payoff. Find the SPNE.



20. Consider this extensive form game. The top number is player 1's payoff, the middle number is player 2's payoff, and the bottom number is player 3's payoff.
    a. Describe the general form of a strategy for each player.
    b. Find the SPNE.



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