Summary

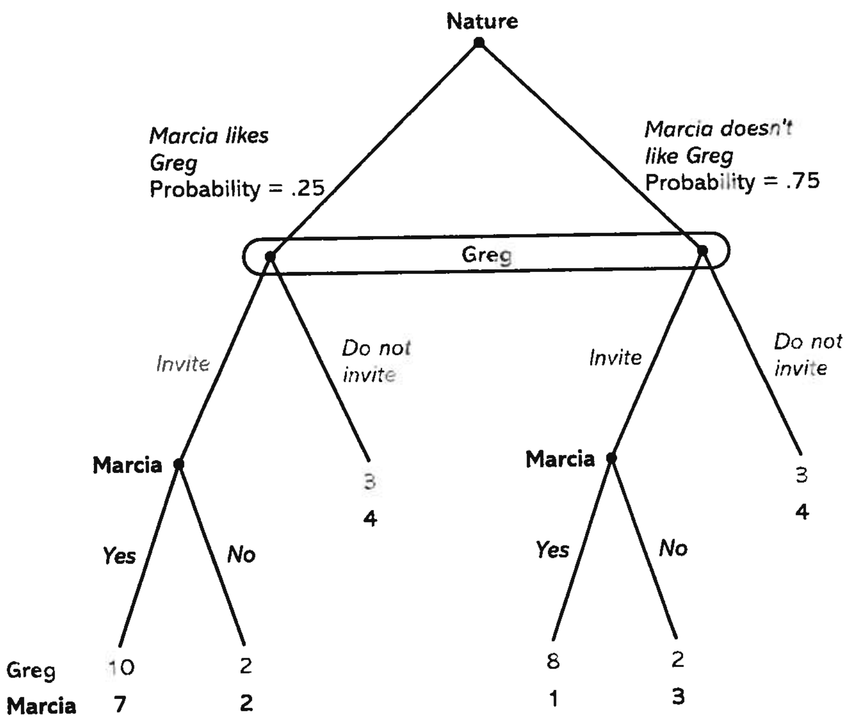
This chapter considers a common and crucial feature of many strategic settings: A person may know something about him- or herself that others do not know. This scenario frequently arises in the form of a player's payoffs being private information. As originally cast, the game is not common knowledge, because, for example, one player doesn't know another player's payoffs. We refer to such a game as having **incomplete information**. The trick to solving a game of incomplete information is to convert it into a game of imperfect information. The initial move in the game is now made by random forces, labeled **Nature**, that determine each player's type, where a **type** encompasses all that is privately known to a player. This Nature-augmented game, which is known as **Bayesian game**, is common knowledge, since, at its start, no player knows his type and thus has no private information. What is commonly known is that Nature will determine players' types. What also is commonly known are the probabilities used by Nature in assigning a player's type.

The solution concept used for Bayesian games in this chapter is **Bayes–Nash equilibrium.** Analogous to Nash equilibrium, it posits that each player's strategy is required to maximize his expected payoff, given other players' strategies, but this definition is supplemented in two ways. First, a player doesn't know other players' types and thus doesn't know what other players will do. However, if a player (accurately) conjectures another player's strategy and has beliefs about the other player's type, he can then form beliefs regarding how another player will behave. Second, given those beliefs, a player's strategy must prescribe an optimal action, and it must do so for *every* possible type of that player.

Examples of equilibrium behavior explored in this chapter included a gunfight in which one gunfighter does not know the skill of the other, negotiations in which a person does not know the ultimate objectives of the person on the other side of the table, committees in which a member does know how informed other committee members are, and auctions in which a person's valuation of the object up for auction is known only to him. We even took on the task of figuring out what to do when threatened by The Joker! These are only a few of the various strategic situations characterized by private information. You face many such scenarios everyday (except the one with The Joker, unless you're Batman).

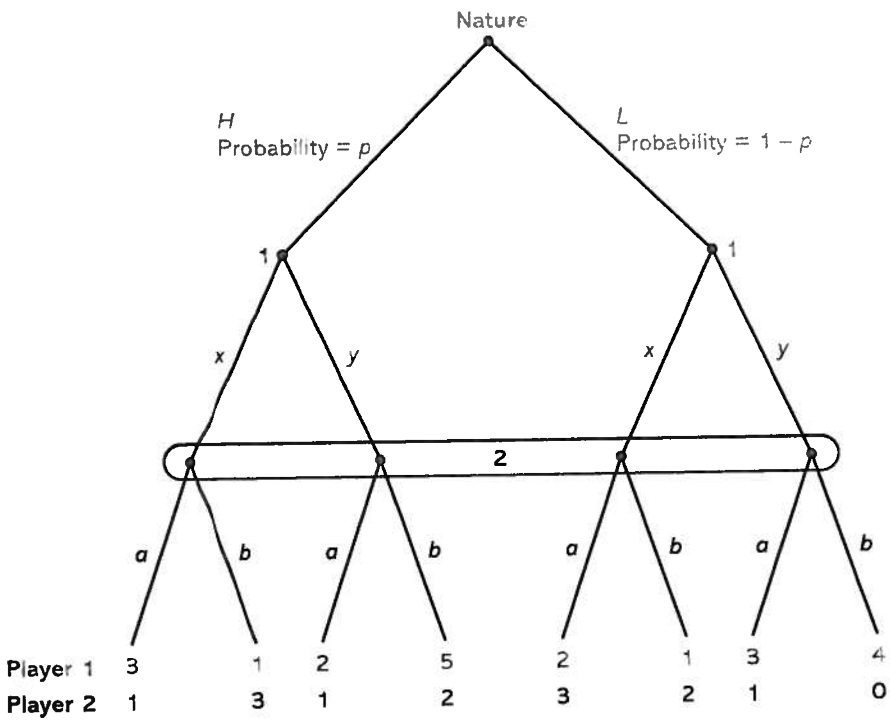
EXERCISES

1. Greg is deciding whether to ask Marcia out on a date. However, Greg isn't sure whether Marcia likes him, and he would rather not ask if he expects to be rejected. Whether Marcia likes Greg is private information to her. Thus, her preferences regarding Greg constitute her type. Greg does not have any private information. Assume that there is a 25% chance that Marcia likes Greg. The Bayesian game is shown here. Should Greg ask Marcia? Find a BNE that answers this question.



2. Consider a gunfight between Bat Masterson and William "Curly Bill" Brocius. Both of the men have private information regarding their skill with a six-shooter. Nature moves first by determining each gunfighter's skill. He can have either a *fast* draw or a *slow* draw. There is a 65% chance that Bat is fast and a 60% chance that Curly Bill is fast. After each gunfighter learns his type—though remaining uncertain about the other gunfighter's type—he chooses between *draw* and *wait*. If both wait, then the payoff is 50. If both draw and (1) they are of the same type (either both fast or both slow), then each has a payoff of 20; and (2) they are of different types, then the fast gunfighter has a payoff of 30 and the slow one of −40. If one draws and the other waits and (1) they are of the same type, then the one who drew has a payoff of 30 and the other a payoff of −40; (2) the one who draws is fast and the other is slow, then the one who drew has a payoff of 30 and the other a payoff of −40; and (3) the one who draws is slow and the other is fast, then each has a payoff of 20. If at least one chooses draw, then there is a gunfight.
   a. Is it consistent with BNE for there to be a gunfight for sure? (That is, both gunfighters draw, regardless of their type.)
   b. Is it consistent with BNE for there to be no gunfight for sure? (That is, both gunfighters wait, regardless of their type.)
   c. Is it consistent with BNE for a gunfighter to draw only if he is slow?

3. Consider a first-price, sealed-bid auction in which a bidder's valuation can take one of three values: 5, 7, and 10, occurring with probabilities .2, .5, and .3, respectively. There are two bidders, whose valuations are independently drawn by Nature. After each bidder learns her valuation, they simultaneously choose a bid that is required to be a positive integer. A bidder's payoff is zero if she loses the auction and is her valuation minus her bid if she wins it.
   a. Determine whether it is a symmetric BNE for a bidder to bid 4 when her valuation is 5, 5 when her valuation is 7, and 6 when her valuation is 10.
   b. Determine whether it is a symmetric BNE for a bidder to bid 4 when her valuation is 5, 6 when her valuation is 7, and 9 when her valuation is 10.
4. Consider a first-price, sealed-bid auction, and suppose there are only three feasible bids: A bidder can bid 1, 2, or 3. The payoff to a losing bidder is zero. The payoff to a winning bidder equals his valuation minus the price paid (which, by the rules of the auction, is his bid). What is private information to a bidder is how much the item is worth to him; hence, a bidder's type is his valuation. Assume that there are only two valuations, which we'll denote L and H, where $H > 3 > L > 2$. Assume also that each bidder has probability .75 of having a high valuation, H. The Bayesian game is then structured as follows: First, Nature chooses the two bidders' valuations. Second, each bidder learns his valuation, but does not learn the valuation of the other bidder. Third, the two bidders simultaneously submit bids. A strategy for a bidder is a pair of actions: what to bid when he has a high valuation and what to bid when he has a low valuation.
   a. Derive the conditions on H and L whereby it is a symmetric BNE for a bidder to bid 3 when he has a high valuation and 2 when he has a low valuation.
   b. Derive the conditions on H and L whereby it is a symmetric BNE for a bidder to bid 2 when he has a high valuation and 1 when he has a low valuation.
   c. Derive the conditions on H and L whereby it is a symmetric BNE for a bidder to bid 3 when he has a high valuation and 1 when he has a low valuation.
   d. Derive the conditions on H and L whereby it is a symmetric BNE for a bidder to bid 1 when he has either a high valuation or a low valuation.
5. Consider this Bayesian game on page 389. Nature chooses the type of player 1, where type H occurs with probability p and type L with probability $1 - p$. Player 1 learns his type and then chooses either action x or action y. Simultaneously, player 2 chooses either action a or action b.
   a. Assume that $p = .75$. Find a BNE.
   b. For each value of p, find all Bayes–Nash equilibria.
6. Two U.S. senators are considering entering the race for the Democratic nomination for U.S. president. Each candidate has a privately known personal cost to entering the race. Assume that the probability of having a low entry cost, f_L, is p and the probability of having a high entry cost, f_H, is $1 - p$. Thus, the type space has just two values. A candidate's payoff depends on whether he enters the race and whether the other senator enters as well. Let



v_2 be a candidate's payoff when he enters and the other senator does as well (so that there are two candidates), v_1 be a candidate's payoff when he enters and the other senator does not (so that there is one candidate), and 0 be the payoff when he does not enter. Assume that

$$v_1 > v_2 > 0,$$
$$f_H > f_L > 0,$$
$$v_2 - f_L > 0 > v_2 - f_H,$$
$$v_1 - f_H > 0.$$

a. Derive the conditions whereby it is a symmetric BNE for a candidate to enter only when she has a low personal cost from doing so.

b. Derive the conditions whereby it is a symmetric BNE for a candidate to enter for sure when she has a low personal cost and to enter with some probability strictly between 0 and 1 when she has a high personal cost.

c. Find some other BNE distinct from those described in (a) and (b).

7. Assume that two countries are on the verge of war and are simultaneously deciding whether or not to attack. A country's military resources are its type, and their relevance is summarized in a parameter which influences the likelihood that they would win a war. Suppose the type space is made up of two values: p' and p'', where $0 < p' < p'' < 1$. A country is type p'' with

probability q and type p' with probability $1 - q$. Consider a country of type p (which equals either p' or p''). If it chooses to attack and it attacks first, then it believes it'll win the war with probability xp, where x takes a value such that $p < xp < 1$. If the two countries both attack, then the probability that a type p country wins is p. If a type p country does not attack and the other country does attack, then the probability of victory for the type p country is yp, where y takes a value such that $0 < yp < p$. Finally, if neither country attacks, then there is no war. A country is then more likely to win the war the higher is its type and if it attacks before the other country. A country's payoff when there is no war is 0, from winning a war is W, and from losing a war is L. Assume that $W > 0 > L$.

a. Derive the conditions for it to be a symmetric BNE for a country to attack regardless of its type.

b. Derive the conditions for it to be a symmetric BNE for a country to attack only if its type is p''.

8. Consider a first-price, sealed-bid auction with three bidders. The payoff to a bidder is 0 when he does not win the item at auction and is the value of the item less his bid (which is the price he pays) when he is the winner. If two or more bidders submit the highest bid, then the winner is randomly determined. Assume that the item has the same value to all three bidders, but they receive different signals as to its value. Nature determines the true value, which is denoted v and can take three possible values: 4, 5, and 6. Each of these values is chosen with probability $\frac{1}{3}$. The signals sent to the three bidders are $v - 1$, v, and $v + 1$; that is, one bidder receives a signal that is too low ($v - 1$), another receives a signal that is too high ($v + 1$), and the third receives a signal that is "just right" (v). Each bidder learns only his own signal, which is the bidder's type. If $v = 4$, then one bidder is given a signal of 3, another bidder is given a signal of 4, and the last bidder is given a signal of 5. If $v = 5$, then one bidder is given a signal of 4, another bidder is given a signal of 5, and the last bidder is given a signal of 6. If $v = 6$, then one bidder is given a signal of 5, another bidder is given a signal of 6, and the last bidder is given a signal of 7. Thus, if a bidder's signal is, say, 5, then he doesn't know if the true value is 4 (in which case he has the highest signal), 5 (in which case he has the accurate signal), or 6 (in which case he has the lowest signal). Given the value, each bidder has an equal chance of receiving one of the three signals. Assume that the minimum increment in bidding is 1, so that the set of feasible bids is {0, 1, . . . , 10}.

a. Show that the following symmetric strategy profile is a BNE:

Signal	Bid
3	4
4	4
5	4
6	4
7	5

b. Show that there is no symmetric BNE in which a bidder's bid is strictly increasing. That is, if $b(s)$ is a bidder's bid, given that her signal is s, and if

$$b(7) > b(6) > b(5) > b(4) > b(3),$$

then this strategy is not a symmetric BNE.

9. *The Newlywed Game* was a popular game show broadcast from 1966 to 1974. On this show, four recently married couples would be queried about one another to see how well each spouse knew the other. Each of the husbands was asked a series of questions while his wife was offstage in a soundproof room. The wives would then return, and each would be asked the same questions. The objective was for the answers of the husband and wife to match. In the second round, they would reverse roles. The couple with the most matched answers would win. Questions asked ran along these lines: What animal does your mother-in-law remind you of? Would your spouse say that your last kiss was ho hum, so so, or ooh la la? In what room of the house does your spouse most like making whoopee? Let us now suppose the husband is asked the question, What drink would best describe your wife on your wedding night: a sloe gin fizz, a Shirley Temple, or a zombie? The husband responds with one of those three choices, and when she is asked the same question, the wife responds with one of the same three choices. If their choices match, a husband–wife pair has a payoff of 100; if they don't match, the pair has a payoff of 0. In casting this as a Bayesian game, suppose the husband's type is which one he believes is true: sloe gin fizz, Shirley Temple, or zombie, where the associated probabilities are $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively. The wife's type is which one she believes is true: sloe gin fizz, Shirley Temple, or zombie. The sloe gin fizz occurs with probability $\frac{1}{2}$, the Shirley Temple with probability $\frac{1}{4}$, and the zombie with probability $\frac{1}{4}$. Players' types are independently drawn.*
   a. Find a BNE.
   b. Find a BNE in which the husband always announces the truth.

10. Players 1, 2, and 3 are involved in a game requiring some coordination. Each chooses among three options: A, B, and C. Nature determines which of these options is the best one to coordinate on, where equal probabilities of $\frac{1}{3}$ are assigned to A, B, and C's being the best one. If all three choose the option that Nature deems best, then each receives a payoff of 5. If all three choose the same option, but it is not the that Nature deems best, then each receives a payoff of 1. If players do not all choose the same option, then each has a zero payoff. Player 1 learns which option is best (i.e., she learns Nature's choice). The three players then simultaneously choose an option. Find a BNE.

11. Consider the Gunfight game when p is the probability the stranger is a gunslinger and $1 - p$ is the probability he is a cowpoke. Assume $0 < p < 1/2$. Find a BNE in mixed strategies in which Wyatt Earp randomizes and the stranger randomizes when he is a cowpoke.

*This is a simplifying, but not very reasonable, assumption, because one would think that if the husband found his wife to be, say, a sloe gin fizz, then it is more likely that the wife found herself to be that way as well.

12. An instructor walks into her classroom and says "What's in your wallet?" No, this is not a Capital One commercial; it's a game (as if it could be anything else, in this book). She selects two students to engage in the following contest. Students simultaneously submit a nonnegative integer. Whoever submits the higher number is to pay an amount in dollars to the instructor equal to the other student's chosen number, and the instructor is to give that student an amount of money in dollars equal to the sum of the money in both students' wallets. The student who submitted the smaller number neither pays nor receives. Let s_i denote the number submitted by student i and w_i be the amount of money in his wallet. According to the rules just described, if $s_1 > s_2$ then bidder 1 wins the contest and receives a net payment of $w_1 + w_2 - s_2$, as the instructor pays him an amount equal to $w_1 + w_2$ and he pays the instructor an amount equal to s_2. In the event of a tie ($s_1 = s = s_2$), each student receives half of what is in each student's wallet and pays half of the number submitted; hence, each student has a net payment of $(w_1 + w_2 - s)/2$. Each student knows how much is in his wallet but does not know how much is in the other student's wallet. A player's type is the amount of money in his wallet and assume it can take any integer from 1 to 100, with each value chosen by Nature with equal probability. Assume a player's payoff equals the amount of money received. Note that this is a common value auction in that the value attached to winning is the same for both students, and students differ in the information they have about that value. Show that it is a symmetric BNE for a student to submit a number equal to twice what is in his wallet.

13. A seller has an object to sell and is deciding at what price to sell it. There is one buyer who values it at \$10 with probability q and \$2 with probability $1 - q$. The seller chooses a price not knowing how much the buyer values it. The price can be any number between 0 and 20. After observing the seller's price, the buyer decides whether or not to buy. The buyer's payoff is zero if she does not buy the item and is the valuation attached to the item less the price paid if she buys it. The seller's payoff equals zero if he does not sell the item and equals the price it sold for if he sells it.

a. Find a BNE in which the buyer's strategy has him act optimally for any price of the seller (and not just the price the seller selects in equilibrium). Assume that when the buyer is indifferent between buying and not buying that she buys.

b. Find a BNE different from that in part (a).

14. Consider two pharmaceutical companies investing in R&D. Each company can invest either at a low (L), medium (M), or high (H) rate. For any pair of investment rates, company 1's profit is known but company 2's profit is private information because only company 2 knows the cost that it incurs. Company 2 either has low cost or high cost. If it is low cost then the payoff matrix is the top figure and if it is high cost then the payoff matrix is the bottom figure. Note that the payoffs for company 1 are the same in the two matrices because its payoff depends only on the investment rates and not on company 2's type. In contrast, company 2's payoffs are higher when it is low cost, holding fixed the investment rates of the two companies. Assume the probability that company 2 is low cost is p and is high cost is $1 - p$. Nature chooses company 2's type and then the two companies simultaneously choose investment rates. Find all BNE.

Company 2 has Low Cost

Company 2

		L	M	H
Company 1	L	7,12	5,15	4,16
	M	9,10	6,12	3,11
	H	8,8	4,9	0,8

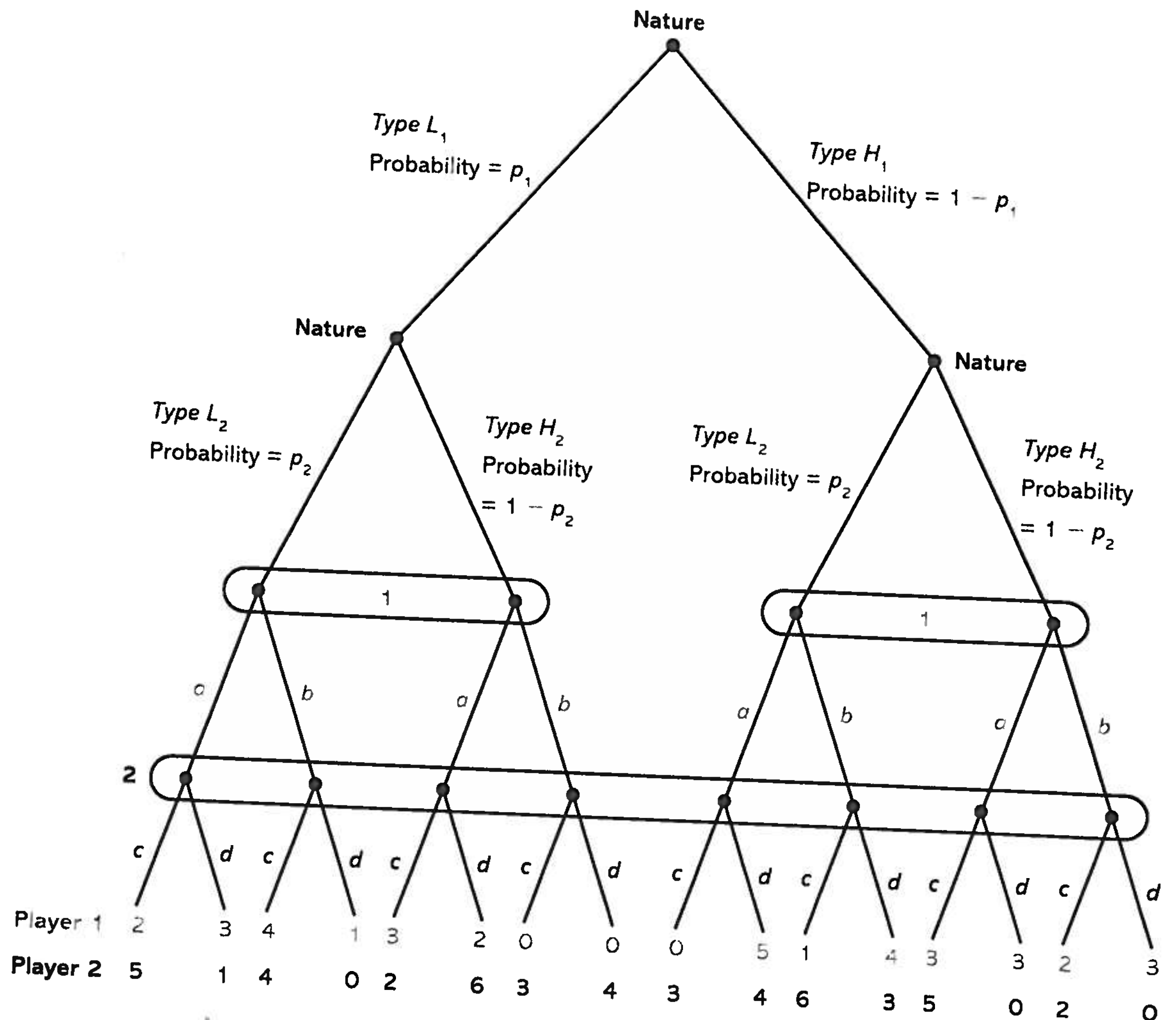
Company 2 has High Cost

Company 2

		L	M	H
Company 1	L	7,3	5,4	4,0
	M	9,2	6,0	3,−4
	H	8,0	4,−3	0,−8

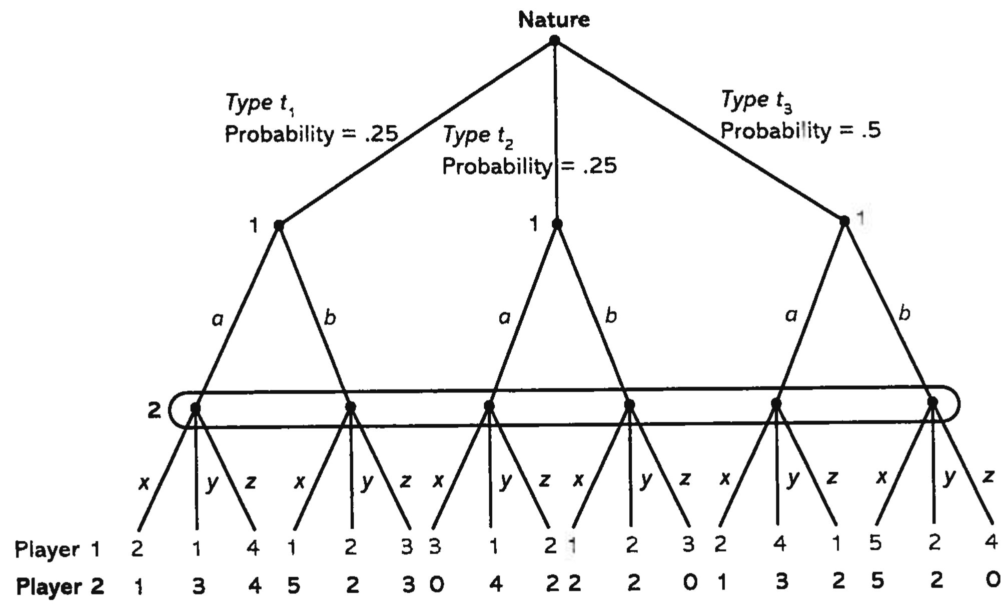
15. Major League Baseball has hired a former NYPD officer as an inspector to keep ballplayers clean of steroids. The inspector has her eye on a particular athlete. Conducting an inspection costs the inspector an amount c and the value to a type i inspector from catching someone on steroids is v_i. There are three types of inspectors who vary in terms of the value of catching a cheater: v_1, v_2, and v_3 where $v_1 > v_2 > v_3 > c > 0$. Thus, if an inspection is conducted by a type i inspector and the athlete tests positive for steroids then the inspector's payoff is $v_i - c$ and if he tests negative then her payoff is $-c$. If no test is conducted then the payoff is zero. As for the athlete, the benefit to him from taking steroids is b_j when he is type j, and the cost of testing positive for steroids is d. There are three types, $b_1 > b_2 > b_3 > 0$, and it is assumed $d > b_1$. The athlete's payoff when he does not take steroids is zero (regardless of whether he is tested for steroids), $b_j - d$ if he took steroids and is caught, and b_j if he took steroids and is not caught. Nature chooses the inspector's type and the athlete's type where each type is chosen independently with probability 1/3. After the inspector learns her type, she decides whether or not to conduct an inspection. After the athlete learns his type, he decides whether or not to take steroids. The inspector and athlete move simultaneously. Find all BNE.

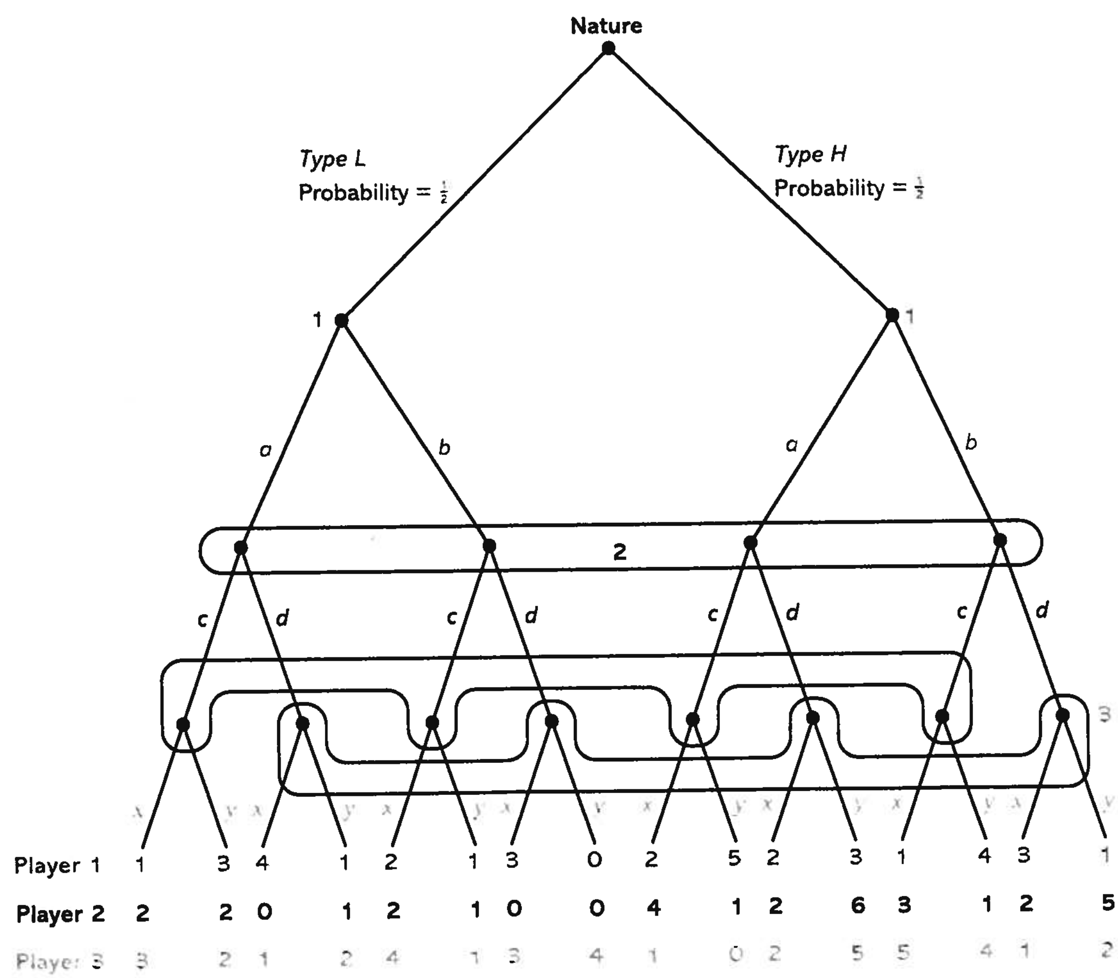
16. Consider the game on page 394. Nature chooses player 1's type which is L_1 with probability p_1 and H_1 with probability $1 - p_1$, and chooses player 2's type, which is L_2 with probability p_2 and H_2 with probability $1 - p_2$. After players learn their type, they simultaneously choose actions; player 1 chooses between a and b and player 2 between c and d.
    a. If player 1 chooses a for either type, derive an optimal strategy for player 2. (Note: Your answer will depend on the values for p_1 and p_2.)
    b. Derive conditions on p_1 and p_2 such that it is a BNE for player 1 to choose a for either type and player 2 to choose c for either type.
    c. Assume $p_1 = .4$ and $p_2 = .3$. Find all of the BNE.



17. Consider the game on the top of page 395. Nature chooses player 1's type which is t_1 with probability .25, t_2 with probability .25, and t_3 with probability .5. After player 1 learns his type, players 1 and 2 move simultaneously with player 1 choosing between a and b and player 2 between x, y, and z. Find all of the BNE.

18. Consider the game on the bottom of page 395. Nature chooses player 1's type which is L with probability 1/2 and H with probability 1/2. After player 1 learns his type, player 1 chooses between actions a and b. Not knowing player 1's type or action, player 2 chooses between c and d. Player 3 observes player 2's choice but does not observe player 1's type or action. Player 3 chooses between x and y. Find all BNE.


Nature
Type t_1 Probability = .25
Type t_2 Probability = .25
Type t_3 Probability = .5
1
1
1
a
b
a
b
a
b
2
x y z x y z x y z x y z x y z x y z
Player 1 2 1 4 1 2 3 3 1 2 1 2 3 2 4 1 5 2 4
Player 2 1 3 4 5 2 3 0 4 2 2 2 0 1 3 2 5 2 0


Nature
Type L Probability = ½
Type H Probability = ½
1
1
a
b
a
b
2
c d c d c d c d
3
Player 1 1 3 4 1 2 1 3 0 2 5 2 3 1 4 3 1
Player 2 2 2 0 1 2 1 0 0 4 1 2 6 3 1 2 5
Player 3 3 2 1 2 4 1 3 4 1 0 2 5 5 4 1 2

Dan and Kristin had their first date last night and each is deciding whether to call the other. Each wants to call only if they both are interested. Dan is then of two types: he likes Kristin (which Nature chooses with probability d) or he does not (with probability $1 - d$). Similarly, Kristin is of two types: she likes Dan (with probability k) or does not (probability $1 - k$). Assume $0 < k < 1$ and $0 < d < 1$. To simplify matters (without any loss of generality), assume that Dan (Kristin) has a choice of whether or not to call only when he (she) likes Kristin (Dan). If a person does not like the other then he or she has no decision. The Bayesian game is shown below. (As a side note, which you can ignore if you don't understand, d is the probability that Dan likes Kristin conditional on Kristin liking Dan, and k is the probability that Kristin likes Dan conditional on Dan liking Kristin. Got it? If not, press the IGNORE button.)

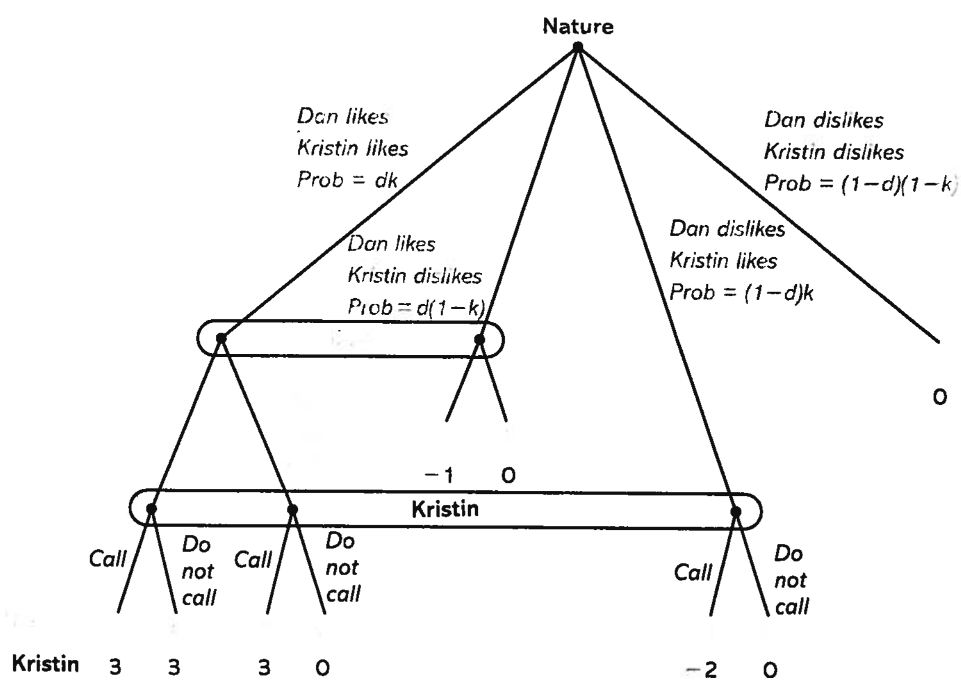
When is it a BNE for Dan to call (when he likes Kristin) but Kristin not to call (when she likes Dan)?

When is it a BNE for Kristin to call but not Dan?

When is it a BNE for both to call?

When is it a BNE for neither to call?

When is it a BNE for them to randomize? Find the mixed-strategy BNE.



20. Consider a market with two firms that are competing in quantities (as modeled in the Appendix to Chapter 6). Inverse market demand is $P = 1 - q_1 - q_2$ where q_i is the quantity of firm i. Each firm has a constant marginal cost of producing. Firm 1's marginal cost is zero so its profit is $(1 - q_1 - q_2)q_1$. Firm 2's marginal cost is c so its profit is $(1 - q_1 - q_2)q_2 - cq_2$. c is private information to firm 2 and equals 0 with probability 1/2 and equals 1/4 with probability $\frac{1}{2}$. Firm 2 is then fully informed, while firm 1 does not know firm 2's marginal cost. Nature chooses the value for c and then firms simultaneously choose quantities. Find the BNE.

10.6 Appendix: Formal Definition of Bayes–Nash Equilibrium

Suppose a game involves simultaneous moves whereby player i selects an action, denoted a_i, from the feasible set of actions for him, which is denoted A_i. Nature moves by choosing each player's type, where player i's type, generically denoted t_i, comes from the set T_i. Each player learns his type (and only his type), and then the players simultaneously choose actions. A strategy for a player assigns an action to each of his possible types.

In choosing an action, a player needs to have beliefs about other players' types. Define

$$t_{-i} = (t_1, t_2, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$$

to be the array of all players' types, excluding player i, and let T_{-i} be the type space for those $n - 1$ players. $\rho_i(t_{-i}|t_i)$ denotes the probability that player i assigns to t_{-i}, conditional on knowing his own type. If players' types are independent random variables, then $\rho_i(t_{-i}|t_i)$ does not depend on t_i. But types may be correlated. This was the case in Section 10.4 in the Common Value and the "Winner's Curse" example, where both bidders' estimates depend on the true value and thus are positively correlated. In that case,

$$\rho_1(s_2|s_1) = \begin{cases} \frac{1}{2} & \text{if } s_2 = s_1 + 4 \\ \frac{1}{2} & \text{if } s_2 = s_1 - 4. \end{cases}$$

With such beliefs over other players' types and a conjecture about other players' strategies, a player can derive beliefs regarding what other players will do. Let $s_j(\cdot)$ denote the strategy of player j. Then $s_j(t')$ is player j's action when $t_j = t'$. The probability that player i assigns to player j's choosing, say, action a' is the probability that player j is a type that chooses a'—that is, the probability that t_j takes a value whereby $s_j(t_j) = a'$. Combining beliefs about another player's type with a conjecture regarding that player's strategy allows the derivation of beliefs about that player's action.

Given the types of the players and the actions selected, let player i's payoff be denoted $V_i(a_1, \ldots, a_n; t_i, t_{-i})$. In many examples, a player's payoff depends only