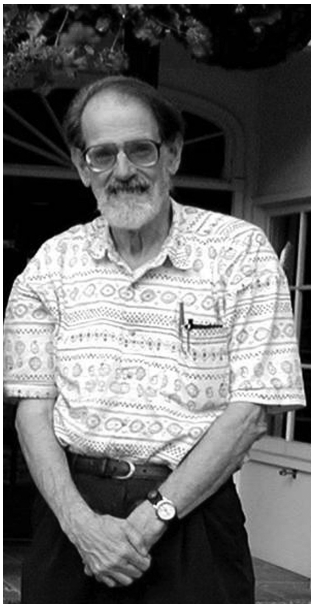
Shapley, Lloyd S. (born 1923)



Lloyd Shapley is considered one of the pioneers of game theory. His most prominent contributions are the inception and study of value theory and core theory. These two theories are the key to solving problems involving the allocation of goods or payoffs achievable through cooperation. Shapley's contributions have led to a broad range of important achievements, such as the exploration of stable solutions for matching and exchange, the measurement of power and a deeper understanding of market economies. His contributions to non-cooperative game theory include the introduction of stochastic games, strategic market games and potential games. Shapley shared with Alvin E. Roth the 2012 Nobel Prize in Economic Sciences.

Lloyd S. Shapley, Professor Emeritus at University of California at Los Angeles (UCLA), was born in 1923 in Cambridge, Massachusetts. He is the son of the renowned astronomer Harlow Shapley. During the Second World War, Shapley served with the US Army Air Corps, and in 1944, he received a Bronze Star for breaking the Japanese and Soviet weather codes. It is said that, after contesting a point John von Neumann made about modelling aerial dogfights, Shapley made such an impression that he was promptly offered a doctoral fellowship at Princeton University (as documented in Leonard (2010)). He completed his PhD in mathematics at Princeton in 1953 under the supervision of Albert Tucker (who, in previous years, also advised John Nash and David Gale). After graduating, Shapley became a research mathematician at the RAND Corporation in Santa Monica, California, from 1954 to 1981. Since 1981 he has been a professor at UCLA, affiliated with both the economics and the mathematics departments.

Shapley is one of the most legendary figures in game theory, making fundamental contributions to the theory of both cooperative and non-cooperative games. Being more a mathematician than an economist, Shapley felt the pull towards more abstract representations of interactive decision problems, creating work that was deep and elegant to purists, but not immediately accessible to many economists. Part of the legend surrounding his work is also due to the fact that many important ideas are to be found in non-traditional publication outlets and unpublished manuscripts.

His voluminous contributions to game theory (and, by extension, microeconomics) are not easy to organise. One logical way to proceed is first to summarise his key developments in the area of cooperative games; then discuss contributions to non-cooperative theory; and finally touch on more fundamental explorations of concepts in individual and multi-person utility.

Cooperative game theory

Most of Shapley's early work was on cooperative game theory. The cooperative approach studies games from an abstract point of view, focusing on the feasible outcomes that can be achieved through cooperation. One looks at the payoffs that can be achieved, rather than the processes that lead to these payoffs. Cooperative game theory deals with questions of how coalitions can form, what coalitions will form and how coalitions that do form divide what they achieve. The theory is key to solving problems of fair and/or stable allocation, but also for guiding the design of solutions that are equitable, defection-proof, and efficient. In the following, some formal concepts will be defined – these will help in understanding Shapley's key contributions to cooperative game theory.

A non-transferable utility (NTU) game in coalitional form consists of a set of players and a payoff possibility set for each subset of players. Such a game becomes a bargaining problem when only the grand coalition (the set of all players) can possibly generate payoff allocations above some exogenously given status quo. Another special case is when utility is transferable, e.g. utility is linear in money and side payments are allowed. The corresponding TU (transferable utility) game is effectively represented by 2^n values, $v(S)$, $S \subseteq N$. Here, $v(S)$ is called the worth of coalition S, interpreted as the maximum value that players of coalition S can jointly achieve by acting in concert. A point solution for a TU coalitional game (N, v) consists of a vector $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ allocating payoff level φ_i to player i such that $\sum_{i=1}^{n} \varphi_i = v(N)$.

Value theory

Shapley's earliest work deals with solutions to TU games and considers the question: what would be an *a priori* valuation of the expected payoff, or the strength, of a player in a coalitional game? The answer to this question is known as the Shapley value. Shapley (1953a) employed an axiomatic approach to derive this solution concept. His four axioms are natural and self-evident: (1) *dummy property* (if a player's contribution to any coalition, $v(S \cup i) - v(S \backslash i)$, is zero, then his value is equal to the worth achievable by himself), (2) *symmetry* (the value does not depend on how players are labelled), (3) *efficiency* (the sum of values over all players is equal to the total worth attainable by the grand coalition) and (4) *linearity* (the value of the sum of two games is equal to the sum of the values of each game). Shapley's remarkable result shows that if a solution has the above set of reasonable properties, then it necessarily takes a specific value, i.e. the Shapley value.

The Shapley value always exists and is unique. It is calculated by taking the expected marginal contribution of each player to a random coalition formed by having the player join in a uniformly distributed random order. Formally, the Shapley value of player i is given by

$$\phi_i^*(v) = \sum_{S \subseteq N \backslash i} \frac{s!(n-s-1)!}{n!}[v(S \cup i) - v(S)],$$

where s and n denote the sizes of coalition S and the grand coalition, respectively.

By tweaking the axioms or considering different setups the Shapley value can be reformulated and generalised. For example, the linearity and dummy axioms can be dropped and replaced by the marginality principle: the value must depend only on the marginal contributions of a player, $v(S \cup i) - v(S \backslash i)$, $S \subseteq N$ (Young, 1985). The Shapley value can also be characterised by symmetry and consistency: if some players are given their Shapley value and dismissed from the game, then what the rest of the players receive in the reduced game does not change (Hart and Mas-Colell, 1989).

The Shapley value has been applied to a wide range of distribution problems: e.g. the structure of multilateral treaties (Bahn *et al.*, 2009), taxation (Aumann and Kurz, 1977), water resource management (Lejano and Davos, 1999), landing fees in airports (Littlechild and Thompson, 1977), fair connection costs in communication and on network games (Bergantiños and Vidal-Puga, 2007; Jackson, 2008), and bandwidth allocation (Niyato and Hossain, 2006).

To illustrate one such application, consider apportioning the costs of a new airport runway. To simplify the example, assume that three airlines, each operating a different type of aircraft, agree to pool resources in building a new runway. Airline A's planes are the smallest of the three, airline B's planes are the next smallest, and airline C's the largest. Clearly, the largest plane determines the size of the runway, so simply dividing the cost of the runway equally is unfair to airlines A and B. However, it would also be unfair for airline C to bear the entire cost, since airlines A and B use the runway too.

The Shapley value takes into account how much each individual or coalition would obtain if it were to act separately and provides a reasonable solution. For this example, assume the costs of each airline building its own runway are as follows: $c(\mathrm{A}) = 6$, $c(\mathrm{B}) = 8$, $c(\mathrm{C}) = 11$. Clearly, if airline B were to build a runway, then it could serve airline A as well, so $c(\mathrm{AB}) = 8$. And if airline C were to build the runway, then it would serve the other two airlines, so $c(\mathrm{AC}) = 11$, $c(\mathrm{BC}) = 11$ and $c(\mathrm{ABC}) = 11$. The Shapley value for this example turns out to be

$$\varphi(c) = (2, 3, 6).$$

This apportioning can be interpreted in a surprisingly commonsensical way – i.e. simply dividing the cost of each segment of the runway equally among the airlines that are able to use that runway segment. The costs of the first, second and third segments are 6, 2 and 3, respectively. Thus, airline A pays for 1/3 of the first segment; airline B pays for 1/3 of the first segment and for 1/2 of the second segment; and airline C pays for 1/3 of the first segment, for 1/2 of the second segment and for the entirety of the third segment.

Sometimes there are no mechanisms, such as fund transfers, that allow the free transfer of utility across players. One natural question is how to generalise the Shapley value to such situations. The literature offers several approaches (e.g. Harsanyi, 1963; Maschler and Owen, 1989). Shapley (1969) himself offered an elegant extension. His λ-transfer method converts a NTU coalitional game into a λ-transferable game. Given a vector $\lambda > 0$ of utility weights, we can provisionally assume that utilities are transferable at rate λ_i/λ_j between players i and j, and calculate the Shapley of such a λ-transferable game. To complete, one looks for a vector $\lambda > 0$ such that the Shapley value of the λ-transferable game involves no transfers (as required by NTU). The method of λ-transfers appears to justify why one should consider solutions for TU games first (Myerson, 1992).

Aumann and Shapley engaged in the challenging problem of extending value theory to the case of a continuum of players. Their work culminated in their 1974 book *Values of Non-Atomic Games*. One remarkable discovery is the 'diagonal principle', by which only the coalitions whose composition constitutes a good sample of the grand coalition matter (e.g. if the population contains 10% of 'engineer types' and 15% of 'physician types', only coalitions containing 10% or engineers and 15% of physicians would matter).

Value theory can be used to measure power in political systems. As early as 1954, Shapley introduced simple games, wherein coalitions have a value of zero or one. A natural interpretation is that of a voting body that requires a minimum number of votes to pass a certain piece of legislation. How much power does a given individual possess? The Shapley–Shubik power index (defined as the Shapley value of the simple game) and the Banzhaf power index, are two possible and related answers (Shapley and Shubik, 1954; Dubey and Shapley, 1979). Power theory has been instrumental in explaining precisely why some actors (such as 'swing votes' in the Supreme Court) wield the power that they do.

In summary, Shapley's (1953a) seminal work has been enormously influential, both through the widespread use of the Shapley value as well as by inspiring other values obtained through modifications to Shapley's original axioms.

Core theory

In addition to notions of strength or fairness, one can consider notions of social stability. Intuitively, a social group is stable if no subgroup can do better on its own by challenging the social order (e.g. separating from the society or taking control over the current order). If the possibilities of each coalition are described by a coalitional game, then the core of such a game is the set of feasible outcomes that cannot be improved upon by any coalition of players. Core allocations are individually rational (no individual acting alone can do better), Pareto efficient (the group of all players acting together cannot do better for all its members) and socially stable (no coalition of players acting on their own means can do better for all its members). The core, in contrast to the value, may be multi-valued or empty. The latter case indicates a natural instability in the game. The core is a powerful concept for analysing allocations (of goods or money) that would or should result from even large numbers of players acting rationally in market or non-market situations. For example, we should not expect a reasonably free, well-functioning and competitive market to produce outcomes that lie outside the core.

This notion of coalitional stability was discussed by Francis Edgeworth, a key figure in the evolution of neoclassical economics, in his 1881 *Mathematical Psychics*. Gillies, in his 1953 dissertation, was the first to formally define the core to examine properties of other solution concepts in TU games (see Gillies, 1969). Shapley, in lectures at Princeton in the fall of 1953, was the first to introduce the core as an independent solution concept for coalitional games.

Over the next two decades, Shapley made fundamental contributions to core theory. The notion of the core allows the study of exchange without involving money or price. Shapley and Scarf (1974) considered an economy with n households, each endowed with one unit of an indivisible object (e.g. a house), and willing to engage in favourable exchanges of one object for another. Households are endowed with ordinal preferences – rankings – over objects. Shapley and Scarf examined the 'top-trading' algorithm proposed by Gale. The idea of the algorithm is as follows. Imagine that traders are nodes in a directed graph. They form edges by each pointing at the owner whose object they most desire (they may be pointing to themselves). If a cycle forms, then each household in a cycle is asked to relinquish his object and take the more desired one. This procedure is repeated with the remaining traders. The process ends after a finite number of steps with an assignment of objects to households. They showed that the resulting allocation could be supported in a competitive equilibrium by a set of prices and hence, by Shapley's theorem, it is in the core (this last result is published by Debreu and Scarf, but credited to Shapley).

Shapley and Shubik (1971) considered a two-sided market in which a product that comes in large, indivisible units (e.g. houses or cars) is exchanged for money, and in which each participant either supplies or demands exactly one unit. They showed that the outcomes in the core of the corresponding TU game are the solutions of a certain linear programming problem dual to the optimal assignment problem. Moreover, these outcomes correspond exactly to the price lists that competitively balance supply and demand.

It is important to ascertain non-emptiness of the core because, whether in real or hypothetical situations (e.g. countries negotiating a treaty over climate change mitigation), an empty core can be expected to lead to stalemate, treaty violations or protracted conflict. Shapley (1967) showed that the core of a game is non-empty if and only if the game is balanced, a result independently established by Bondareva in 1963. For the NTU case, Shapley (1973) offered an alternative proof of Scarf's theorem that a balanced game always has a non-empty core. His proof involves an elegant extension of the Knaster–Kuratowski–Mazurkiewicz (KKM) Theorem, currently known as the KKMS Theorem. Shapley (1971) introduced convex games, i.e. TU games in which the worth of a coalition increases rapidly with its size. These games always possess non-empty cores and the centre of gravity of the extreme points of the core is the Shapley value.

Shapley and Shubik (1969) studied the possibility of representing coalitional games by markets. They show that a TU game is representable by a market if and only if the game is totally balanced (games whose subgames possess non-empty cores). The markets they considered were pure exchange economies with money. Later, Billera (1974) established a version of this result for markets with production but without money. Shapley and Shubik (1975) proved that for any TU market game, the direct market represents the game and that its competitive payoff vectors completely fill up the core. Furthermore, given any point in the core, there exists a representing market that has the given core point as its unique competitive payoff vector. Shapley and Shubik conjectured that the same result holds with respect to the inner core for NTU games, a claim later verified by Qin (1993).

Matching

A cooperative game has 2 to the power of n coalitions and, hence, it is difficult to check if the core is non-empty. The situation is simplified if individual improvements can be possibly made only by forming pairs. For any given allocation, one only needs to check that no pair of two players can do better separately. For example, consider a roommate problem in which four players, A, B, C and D, need to create two pairs in order to share two rooms. The preferences of individuals with respect to whom they would like to share a room are:

A: $B > C > D$,
B: $C > A > D$,
C: $A > B > D$, and
D: any preference.

Let us begin with the matching (AB, CD). This matching is not stable because both B and C can be better off forming a pair than staying with their current partners. But the new matching, (BC, AD), is not stable because A and C are better off forming a pair than staying with their current partners. However, (AC, BD) is unstable because A and B have incentives to form a pair, thus returning to (AB, CD). Because these are the only three possible arrangements, this matching problem does not possess a stable solution.

Gale and Shapley (1962) considered the stability of two-sided matching. There are two types of players and value is created only when pairs have one individual from each type. A situation that naturally comes to mind is a set of boys and girls who need to be matched to form heterosexual couples. The essential data is nothing more than a double list of ordinal rankings, where each player ranks the members of the opposite sex in order of their desirability to him or her. Gale posed the question: does a stable arrangement exist for the two-sided matching problem? Shapley settled the question in the positive by discovering the now famous deferred-acceptance algorithm.

The deferred-acceptance algorithm works as follows. Let each boy make a matching proposal to his most preferred girl. Each girl who receives more than one proposal rejects all the boys except for the one she prefers the most. Importantly, each girl does not yet accept the proposal that she holds on to, but waits until the end of the algorithm (deferred-acceptance). In the second round, each of the rejected boys makes a proposal to their second-ranked girl. Again, each girl receiving new proposals rejects all but the one she prefers the most, and so on. The algorithm ends when no girl rejects any proposal, at which point all proposals are accepted and a matching is completed. This happens after a finite number of steps, since no boy makes a proposal twice to the same girl. The matching is stable because no boy can switch to a more preferred girl who would accept him (he has been rejected by all the girls who rank ahead of the one he gets matched to).

By the properties of the algorithm, a stable matching exists and it is reached in a finite number of steps. Moreover, it is optimal and strategy-proof for the side that proposes (Gale and Shapley, 1962; Roth and Sotomayor, 1992). In the concluding remarks of the 1962 paper, the authors revel in the fact that ‘The argument is carried out not in mathematical symbols but in ordinary English; there are no obscure or technical terms'. They also predicted that the algorithm would eventually be applied in practice.

The matching solution applies to many real-world economic situations where goods are lumpy or indivisible and when there are no price or market mechanisms available to assign goods. Shapley's work on matching markets in the early 1960s with non-transferable utility and in the early 1970s with transferable utility led to an enormous volume of work by many others, most notably Alvin E. Roth, on a large variety of matching markets: assigning residents to hospitals, pairing kidney donors and transplant patients, assigning teenagers to public high schools, and many others. Roth and Shapley shared the 2012 Nobel Prize for the theory of stable allocations and the practice of market design.

Noncooperative game theory

Strategic equilibrium

In addition to value theory and core theory, Shapley is also an important figure in non-cooperative game theory. John Nash, whose equilibrium concept offers a fundamental solution for non-cooperative games, and Shapley were close associates in Princeton. Anecdotally, Nash originally had thought of calling his solution a stable point, but Shapley convinced him to go with 'equilibrium' instead (i.e. Nash equilibria can often be unstable, like a marble on top of a mountain peak) (as recounted by Shapley to one of the authors).

Shapley (1953b) introduced stochastic games to model the dynamics of ongoing non-cooperative game situations in which the game changes over time as a function of the players' strategic choices and the state of nature. Shapley (1953b) characterised the minimax value of zero-sum stochastic games and showed that the equilibrium can be supported by strategies that depend only on the state.

Aumann and Shapley (1976) were the first to produce a version of the 'folk' theorem, a central result in the theory of repeated games. Their theorem states that every feasible and individually rational payoff in a game can be supported by a subgame perfect equilibrium of the undiscounted infinitely repeated game. Later versions of the theorem employ discounted payoffs. These theoretical contributions to stochastic games and repeated games are now recognised as the key to understanding how bargaining in real economic situations occur. The theory has also proven valuable in other areas such as evolutionary biology and artificial intelligence.

Fundamental to the general (Walrasian) model of market exchange is the requirement that transactions be governed by a uniform price system. That is, the law of one price is imposed in the general equilibrium model. However, prices in the general equilibrium model are given *ex machina* and are not responsive to agents' buying and selling decisions. Agents are passive with respect to prices at which they trade. A theory is therefore needed to account for how prices get formed. Shapley and Shubik (1977a) offered a model, a.k.a. the trading post model, that is strategically closed, in that prices are determined by decisions of agents and the system as whole responds meaningfully to agents' decisions. A good summary of the trading posts model can be found in Mas-Colell Whinston and Green (1995, Example 18.C.3). The trading post model has applications in a wide range of areas, including contemporary monetary macroeconomics with endogenous demand for money (see, for example, the special issue on strategic market games in the *Journal of Mathematical Economics*, vol. 39, issue 5–6 of 2003).

Utility theory

Finally, Shapley made important contributions to utility theory. He provided foundations of cardinal utility based on intensity of preferences. Shapley developed expected utility theory without the completeness axiom, which was later used as foundation for multi-person utility (Baucells and Shapley, 2008).

In his class notes at UCLA, Shapley emphasised the fact that each solution concept has attached to it a utility category. Specifically, if we arrange the utilities of the players in a vector, every solution concept is invariant to a certain family of transformations of this utility vector. The largest group of transformations defines the utility category of the solution concept (Moulin, 1991). For example, mixed strategy equilibrium is invariant to affine transformations of each individual utility. Hence it is a cardinal solution concept.

The challenge is either to discover a solution concept belonging to a larger utility category, or to provide an impossibility result. Shapley did both. For two-player bargaining situations, Shapley (1969) showed that no ordinal solution concept exists (recall that Nash bargaining solution allows only the linear scaling of individual utilities). For three-player bargaining situations, Shapley was able to define a solution concept that exits, is unique, and is invariant to order-preserving transformations. The solution, which Shapley never formally published, is informally known as 'Quaker oats' (see Shubik, 1982, section 4.3.3).

Shapley named this bargaining solution after the Quaker oats container that shows a Quaker holding a Quaker oats container that shows a Quaker holding a Quaker oats container, etc. Consider a Pareto surface and the 'triangle' formed by its intersection with the disagreement planes. We define a second 'triangle' as the largest possible triangle inside the first triangle such that each edge follows the indifference curve of one of the three players. This procedure can be repeated *ad infinitum*, and converges to a unique point in the Pareto surface. As with Walrasian equilibrium, the procedure is entirely based on indifference curves, and hence belongs to the ordinal class. It was not until the 2000s that ordinal solutions in the presence of four or more players were found (Samet and Safra, 2005). They are all based on Shapley's Quaker oats solution.

Concluding remarks

Some of Shapley's important work did not come to light in the usual way. Defying traditional academic convention, much of his writings took the form of RAND reports, book chapters, notes for his classes at UCLA and unpublished material. His doctoral students (authors of this entry included) can attest to a bag of papers that Shapley would often carry, containing gems of ideas, many unpublished. Some of his unpublished contributions have been acknowledged or published by others who needed to build on these results (Shubik, 1982). For instance, Aumann (1962) credited Shapley for the independent formulation of expected utility without the completeness axiom. Debreu and Scarf (1963) credited Shapley for the proof that the Walrasian equilibrium produces outcomes in the core. And the Shapley–Folkman theorem, an important result in convex geometry, appears as an appendix in Starr (1969).

Two themes repeat in Shapley's work: constructive algorithmic proofs and clever counterexamples. We have already discussed the 'deferred-acceptance' algorithm in matching. Shapley (1974) provided an alternative, constructive proof of the existence of Nash equilibrium in two-person non-cooperative games applying the path-following Lemke–Howson algorithm. Each equilibrium point has an orientation: there is always one more 'negative' than 'positive' equilibrium point. While not all equilibria are accessible, Shapley (1981) examined how to transform problems so that previously inaccessible solutions become accessible by means of path-following algorithms. Monderer and Shapley (1996) introduced potential games and showed that these games have a pure-strategy equilibrium by following a finite improvement path. Shapley was awarded the Informs John von Neumann Theory Prize in 1981 for his contributions to operations research.

Shapley is also famous for providing counterexamples. Shapley and Shubik (1977b) constructed an economy with quasi-linear utilities and three equilibria, showing that the conditions for uniqueness of equilibrium cannot be relaxed much. For a while it was an open question whether fictitious play converges to equilibrium. In fictitious play, two automatons play a repeated game by choosing the best response to the frequency of play employed by the opponent during the previous rounds. Shapley (1964) constructed a 'rock–scissors–paper'-like game in which fictitious play cycles over the three strategies. Moreover, the frequency of the cycles increases with the number of plays, hence failing to converge to the mixed strategy equilibrium. Another instance is a 21-player counterexample he constructed, with Kikuta (1986), to resolve a longstanding question in cooperative game theory.

Shapley is well known as an engaging wit and a rare intellect. On the association of his name with the value, he said: 'I seem to be turning into an adjective. I would still like to be a noun' (http://articles.economictimes.indiatimes.com/2003-01-17/news/27560412_1_game-theory-john-f-nash-harvard). His far-ranging work, which has left its imprint on almost every aspect of game theory, is perhaps summed up best by Robert Aumann who wrote:

'Shapley's work in Game Theory – both applied and mathematical – is truly astounding in scope, in depth, in beauty, and in importance. On each of these counts, Shapley has done more than all the previous Game Theory Nobelists, even when taken together.' (http://www.econ.ucla.edu/news/shapley/)

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See also

Aumann, Robert J. (born 1930);
bargaining;
Gale, David (1921–2008);
games in coalitional form;
game theory;
Harsanyi, John C. (1920–2000);
matching and market design;
Nash, John Forbes (born 1928);
Nash equilibrium;
Roth, Alvin (born 1951);
Shapley value

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